EXTENSIONS OF THE ERDŐS-KO-RADO THEOREM TO PERFECT MATCHINGS

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Mahsa Nasrollahi Shirazi

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SUPERVISORY AND EXAMINING COMMITTEE

Mahsa Nasrollahi Shirazi, candidate for the degree of Doctor of Philosophy in Mathematics**,** has presented a thesis titled, ***Extensions of the Erdős-Ko-Rado Theorem to Perfect Matchings,*** in an oral examination held on February 25, 2022. The following committee members have found the thesis acceptable in form and content, and that the candidate demonstrated satisfactory knowledge of the subject material.

External Examiner:	*Dr. Krystal Guo, University of Amsterdam
Co-Supervisor:	*Dr. Karen Meagher, Department of Mathematics & Statistics
Co-Supervisor	*Dr. Shaun Fallat, Department of Mathematics & Statistics
Committee Member:	*Dr. Allen Herman, Department of Mathematics & Statistics
Committee Member:	*Dr. Sandra Zilles, Department of Computer Science
Committee Member:	*Dr. Shahla Nasserasr, Department of Mathematics & Statistics
Chair of Defense:	*Dr. Harold Weger, Department of Biology

*via ZOOM Conferencing

Abstract

One of the important results in extremal set theory is the Erdős-Ko-Rado (EKR) theorem which gives a tight upper bound on the size of intersecting sets.

The focus of this thesis is on extensions of the EKR theorem to *perfect matchings* and *uniform set partitions*. Two perfect matchings are said to be t-intersecting if they have at least t edges in common. In 2017, Godsil and Meagher algebraically proved the EKR theorem for intersecting perfect matchings on the complete graph with $2k$ vertices [16]. In 2017, Lindzey presented an asymptotic refinement of the EKR theorem on perfect matchings [21]. In this thesis, we extend their results to 2-intersecting and also to set-wise 2-intersecting perfect matchings. These results are not asymptotic. A perfect matching is in fact a special case of a *uniform set partition*. Another focus of this thesis is on partially 2-intersecting uniform set partitions. We find the largest set of 2-intersecting uniform set partitions, when the number of parts is sufficiently large. The result on uniform set partitions is part of a joint research project with Karen Meagher and Brett Stevens.

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The results in chapter 5 of my thesis were mainly from a joint work with Karen Meagher and Brett Stevens. Brett also introduced me to the clique used in Theorem 3.3.4. I wish to thank Brett for his valuable discussions, comments which greatly improved some of the results in my thesis.

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Chapter 1

Introduction

In 1961, Erdős, Ko and Rado made an early and fundamental contribution to the field of extremal combinatorics with the following result, which has become known as the EKR theorem.

Theorem 1.0.1 (EKR)**.** *[9] If $\mathcal{F}$ is a t-intersecting family of k-subsets of $\{1, 2, \ldots, n\}$, then there exists a function $f(k, t)$ such that if $n \geq f(k, t)$, then*

$$|\mathcal{F}| \leq \binom{n-t}{k-t}.$$

If equality holds, then $\mathcal{F}$ consists of all k-subsets containing a fixed t-subset of $\{1, 2, \ldots, n\}$.

This result has motivated consideration for study other "intersecting" families of many other combinatorial objects using diverse proof techniques and has developed into an active and broad area of research. There are many recent results giving analogs of the EKR theorem for different mathematical objects, such as permutations [14, 19], uniform set-partitions [23], t-designs [26], vector spaces [11], and perfect matchings [16, 21].

Twenty-three years after the publication of Erdős, Ko and Rado's work, Wilson [35] enhanced the impact of these results by giving an algebraic proof of their result with the exact value of $f(k,t)$ for all k and t. Later in 1997, Ahlswede and Khachatrian [1] found all maximum t-intersecting families of k-subsets for all values of n. In 2011, Ellis, Friedgut, and Pilpel [7] showed that the analog of the EKR theorem holds for t-intersecting families of permutations of $\{1,\ldots,n\}$, when n is sufficiently large relative to t.

1.1 EKR theorem for intersecting perfect matchings

In 2005, Meagher and Moura [23] proved that a natural version of the EKR theorem holds for uniform set-partitions. Note that a perfect matching is a special case of a uniform set-partition in which each block is of size 2. but their proof did not apply very well to perfect matchings. Recently, an algebraic proof of this well-known theorem for intersecting families of perfect matching was found by Godsil and Meagher [16] and their proof is based on eigenvalue techniques originally utilized by Wilson [35]. Further, they conjectured a version of the EKR theorem holds for t-intersecting families of perfect matchings, when $2k \geq 3t+2$. In 2018, Lindzey [21] proved this conjecture for all t, provided that k is sufficiently large relative to t, but his proof does not give any lower bound on how large k must be. In this thesis we prove the conjecture holds for $t = 2$ and all $k \geq 3$. In Chapters 3 and 4 we present some new results on the extensions of the EKR theorem for (set-wise) intersecting perfect matching. The approaches we take are inspired by certain eigenvalue techniques utilized first by Wilson [35], and then modified by Godsil and Meagher [16]. The results of the Chapter 3 appear in the paper [10] written by Fallat, Meagher, and the author, and the results of the Chapter 4 are included in the paper [30] by the author.

1.2 EKR theorem for intersecting uniform set-partitions

In [9], Erdős and Székely considered different types of intersection for partitions. One of these types, and the one we consider here is defined as: two partitions P and Q are *intersecting in a pair* if there exist blocks P_i in P, and Q_j in Q such that $|P_i \cap Q_j| \geq 2$. Their work considers all partitions, not just uniform partitions.

In [23], Meagher and Moura generalized this definition: two partitions P and Q are *partially t-intersecting* if there exist P_i in P, and Q_j in Q such that $|P_i \cap Q_j| \geq t$. This work is different than that of Erdős and Székely since only uniform partitions are considered in [23]. In their work, they present a conjecture on a generalization of the EKR theorem for uniform set-partitions. In Chapter 5, we will prove an asymptotic version of their conjecture. This work is a first movement on this conjecture in almost

20 years. The results of Chapter 5 are included in the paper [24] written by Meagher, Stevens, and the author.

1.3 Overview of the document

In Chapter 2, we introduce the general concept of uniform set-partitions. A special example of uniform set-partitions are perfect matchings. First we explore perfect matchings on complete graphs with an even number of vertices; later we introduce some different types of intersections on uniform set-partitions. We present our first result on the extension of the EKR theorem in Chapter 3. We prove the EKR theorem on 2-intersecting perfect matchings of the complete graph on $2k$ vertices; for all values of k. First we change the problem of finding the maximum size of an intersecting set of perfect matchings into the problem of finding a maximum coclique in a special graph; which is a graph in the perfect matching association scheme. We also introduce the Delsarte-Hoffman bound as a tool to determine the size of the maximum clique. Later we gather some information on the eigenvalues of the aforementioned graph which requires some important details regarding the character table of the perfect matching association scheme to be discussed. In general finding the character table of the perfect matching association scheme for $k \geq 41$ is still an open problem. Using the concept of quotient graphs, we construct a portion of the character table of the perfect matching association scheme. All of this information together helps to prove the EKR theorem for 2-intersecting perfect matchings on the complete graph on $2k$ vertices. Then in Chapter 4, we prove a version of the EKR theorem for set-wise 2-intersecting perfect matchings, which is a new type of intersection on perfect matchings. This result is a generalization of the previous results in Chapter 3, and the approach we take is similar. The focus of Chapter 5 is on 2-intersecting uniform set-partitions. We prove that for k sufficiently large, the set of all uniform set-partitions with k blocks of size ℓ, in which every block contains a fixed pair is the largest set of partially 2-intersecting uniform set-partitions. At the end, in Chapter 6, we present a list of interesting open problems and conjectures derived from the research work involved with my doctoral studies.

Chapter 2

Partitions

One of the important problems in combinatorics is to find the size of a largest family of sets which has a special structure and some given constraints on the observed set system. This is the main idea behind the celebrated Erdős-Ko-Rado theorem and its extensions.

In this chapter we introduce the concept of uniform set partitions. In Sections 2.2 and 2.3 we explore perfect matchings of the complete graph K_{2k} as a special case of uniform set partitions. In Chapters 3 and 4, perfect matchings are the main object considered for developing a version of the Erdős-Ko-Rado theorem. Later, in Section 2.4, we define some different types of structures and constraints (these can be considered to be different types of intersection) on uniform set partitions to provide the ground work for some interesting results in the coming chapters.

2.1 Uniform Set Partitions

A *set partition* $P = \{P_1, \dots, P_k\}$, of the set $\{1, 2, \dots, n\}$ is a grouping of the elements of this set into nonempty subsets (parts or classes) such that each element is included in exactly one part. Throughout this thesis, a set partition is referred to as a *partition* for simplicity. A partition P is called a *k-partition* if it includes k parts, which is denoted by $|P| = k$. A partition in which all parts have even size is called an *even partition*. An *integer partition* of a positive integer n is a list $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_k]$ of positive integers with $n = \sum_i \lambda_i$; typically in an integer partition we assume $\lambda_i \geq \lambda_{i+1}$. We denote an integer partition of n by $\lambda \vdash n$. An integer partition in which all parts have even size is called an *even partition*. For example, if $\lambda \vdash n$, with $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_k]$, then we define $2\lambda = [2\lambda_1, 2\lambda_2, \dots, 2\lambda_k]$ to be an even partition of $2n$. The set of sizes of the parts in a set partition P of $\{1, 2, \dots, n\}$ is an integer partition of n, we call this integer partition the *shape* of P. (See [15, Sections 3.1 and 15.4] for more details about partitions.)

A (ℓ, k)-*partition* is a set partition of $\{1, 2, \dots, k\ell\}$ with exactly k blocks each of size ℓ. These are also called *uniform set partitions*. We use $\mathcal{U}_{\ell,k}$ to denote the set of all (ℓ, k)-partitions, and $u_{\ell,k} = |\mathcal{U}_{\ell,k}|$. It is easy to see that

$$u_{\ell,k} = \frac{1}{k!}\binom{k\ell}{\ell}\binom{k\ell - \ell}{\ell}\binom{k\ell - 2\ell}{\ell}\cdots\binom{\ell}{\ell}. \tag{2.1}$$

A particular class of uniform set partitions are $(2, k)$-partitions on a set of size $2k$. For the complete graph K_{2k}, such a partition on the set of vertices forms a set of edges in which every vertex is covered exactly once. This $(2, k)$-partition is called a *perfect matching*. In Sections 2.2, and 2.4, the concept of perfect matchings will be discussed in more detail.

2.2 Background On Perfect Matchings

A *matching* M in a graph X is a set of edges such that no two edges have a vertex in common. If a matching covers every vertex of X, it is called a *perfect matching* [17]. In this thesis we mostly consider perfect matchings in complete graphs with an even number of vertices.

It is easy to check that the number of perfect matchings in K_{2k} is

$$\frac{1}{k!}\binom{2k}{2}\binom{2k-2}{2}\cdots\binom{2}{2} = (2k-1)(2k-3)(2k-5)\cdots 1.$$

For any positive integer k, we define

$$\begin{aligned} (2k-1)!! &:= (2k-1)(2k-3)(2k-5)\cdots 1, \\ (2k)!! &:= (2k)(2k-2)(2k-4)\cdots 2. \end{aligned}$$

This definition agrees with the usual definition of double factorial. The number of perfect matchings in K_{2k} is $(2k-1)!!$.

2.3 Perfect Matching Association Scheme

In this section, we first provide the reader with some well-known terminology and some concepts in graph theory which will be needed later throughout this thesis. For more details we refer the reader to the books [15, 17].

Let X be a graph. A *clique* in X is a set of vertices in which any two are adjacent; a *coclique* is a set of vertices in which no two are adjacent. The size of a largest clique and a largest coclique are denoted by $\omega(X)$ and $\alpha(X)$, respectively. The *adjacency matrix* $A(X)$ of X is a matrix in which rows and columns are indexed by the vertices and the (i, j)-entry is 1 if $i \sim j$, and 0 otherwise. A *weighted adjacency matrix* $A_W(X)$ subordinate to X is a symmetric matrix in which rows and columns are indexed by the vertices and the (i, j)-entry may be non-zero (which is interpreted as its edge weight) if $i \sim j$ and is 0 otherwise. The *eigenvalues* of X refer to the eigenvalues of its adjacency matrix. We use $\mathbf{1}$ to denote the all-ones vector; for any d-regular graph, the all-ones vector is an eigenvector with eigenvalue d.

An *association scheme* is a set of graphs which are highly structured. Most of the graphs we consider are part of an association scheme. It is easier to define an association scheme in terms of matrices. In this thesis, we follow [15, Section 3.1] for definition of association scheme, as definitions vary across the field.

Definition 2.3.1. *An association scheme is a set of* $v \times v$ *matrices* $\mathcal{A} = \{A_0, \ldots, A_d\}$, *such that,* *(i)* $A_0 = I$,

(ii) $\sum_{i=0}^{d} A_i = J$,

(iii) $A_i^T \in \mathcal{A}$, *for all* $i = 1, \ldots, d$

(iv) *for all* $i, j \in \{0, \ldots d\}$, *the product* $A_i A_j$ *lies in the span of* $\mathcal{A}$,

(v) $A_i A_j = A_j A_i$ *for all* i *and* j.

Here the matrices I *and* J *refer to the identity and the all-one matrices both of size* $v \times v$, *respectively.*

Further, if for all i, $A_i^T = A_i$, we say the association scheme is *symmetric*. It is easily seen that any perfect matching of K_{2k} is an even set partition with all parts of size 2; and the shape of any perfect matching is the integer partition $[2, 2, \ldots, 2]$. Taking the union (or overlapping) of two perfect matchings in K_{2k} produces a set of disjoint even cycles in K_{2k}, where the union of parallel edges gives rise to 2-cycles. Any two perfect matchings in K_{2k} produce a set partition of the set $\{1, \ldots, 2k\}$ where each part is the set of vertices contained in one of the even cycles in the union of the two perfect matchings. The shape of this set partition is the integer partition $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_\ell]$, here the cycles have size λ_i. Such an integer partition will be even. Using this we define a set of graphs on the collection of all perfect matchings.

Definition 2.3.2. *Let* k *be an integer and* $\lambda \vdash k$. *Define* $A_{2\lambda}$ *to be a matrix in which the rows and columns are indexed by perfect matchings of* K_{2k}. *In* $A_{2\lambda}$ *the*

(P, Q)-entry is 1 if the union of the perfect matchings P and Q has shape 2λ, and 0 otherwise.

We let

$$\mathcal{A} = \{A_{2\lambda} \mid \lambda \vdash k\}.$$

The set $\mathcal{A}$ forms a symmetric association scheme which is known as the *perfect matching association scheme* [15, Section 15.4], and the matrix algebra $\mathbb{C}[\mathcal{A}]$ constructed by the complex linear combinations of the matrices in $\mathcal{A}$ is called the *Bose-Mesner algebra* of this association scheme. For a matrix $A_{2\lambda} \in \mathcal{A}$ define the graph $X_{2\lambda}$ so that $A_{2\lambda}$ is its adjacency matrix. A *graph in the association scheme* is any graph X with $A(X) \in \mathbb{C}[\mathcal{A}]$. Every graph in this association scheme is an undirected graph with the set of perfect matchings as its vertex set.

We state some properties of this association scheme and refer the reader to [15, Chapter 3] for more details and proofs. Denote the *Schur product* of two matrices $A = [a_{i,j}]$ and $B = [b_{i,j}]$ of the same size by the matrix $A \circ B$ whose (i, j)-entry is given by $[a_{i,j}b_{i,j}]$. For any two matrices A_{λ_i} and A_{λ_j} in $\mathcal{A}$,

$$A_{\lambda_i} \circ A_{\lambda_j} = \delta_{i,j} A_{\lambda_i}.$$

This implies the matrices A_{λ_i} are linearly independent and Schur orthogonal, meaning that their Schur product is the zero matrix. Hence the matrix set $\{A_{2\lambda} \mid \lambda \vdash n\}$ is an orthogonal basis for the Bose-Mesner algebra of this scheme. Further, the matrices in $\mathcal{A}$ are symmetric and commute, thus they are simultaneously diagonalizable. The *character table* of the perfect matching association scheme is a matrix such that its (i, j)-entry is the eigenvalue corresponding to the ith eigenspace of the jth class (matrix A_{λ_j}).

The group $\mathrm{Sym}(2k)$ acts transitively on the set of perfect matchings of K_{2k}, and the stabilizer of a single perfect matching under this action is isomorphic to the wreath product $\mathrm{Sym}(2) \wr \mathrm{Sym}(k)$. The action of the group $\mathrm{Sym}(2k)$ on the set of perfect matchings is equivalent to its action on the cosets of $\mathrm{Sym}(2k)/\left(\mathrm{Sym}(2) \wr \mathrm{Sym}(k)\right)$. The perfect matching scheme is the Schurian association scheme, or the orbital scheme of the action of $\mathrm{Sym}(2k)$ on the cosets $\mathrm{Sym}(2k)/\left(\mathrm{Sym}(2) \wr \mathrm{Sym}(k)\right)$. This group action gives a representation of $\mathrm{Sym}(2k)$. We will use representation theory, following [28].

In the character table of the perfect matching association scheme, rows correspond to irreducible modules and the eigenspaces are unions of modules. We note that the $(2k-1)!!$-dimensional vector space of vectors indexed by the perfect matchings is a $\mathrm{Sym}(2k)$-module. It is well-known that the irreducible representations of $\mathrm{Sym}(2k)$ correspond to integer partitions of $2k$ [27] and that this module can be expressed as the sum of irreducible modules $\mathrm{Sym}(2k)$ corresponding to even integer partitions of $2k$ [29]. We denote these irreducible modules by the corresponding even integer partitions (see [15, Chapter 15] for details). (For example, $[2k]$ will be used to denote the irreducible module corresponding to the trivial representation; this is the

1-dimensional vector space of constant vectors of length $(2k-1)!!$.) It follows that the common eigenspaces of the matrices in the perfect matching association scheme are unions of these irreducible modules, thus the common eigenspaces in the perfect matching association scheme correspond to the even integer partitions of $2k$. Further details on these eigenspaces are presented in Subsection 3.2.2.

Another example of an association scheme is the *Johnson scheme* which will be used later in the proof of Theorem 3.2.14.

Definition 2.3.3. *[15, Chapter 3] For $2k \leq n$, define $J(n,k,i)$ to be the graph in which the vertices are the k-subsets of a fixed set of size n. In this graph two vertices P and Q are adjacent if $|P \cap Q| = k-i$. The set of all graphs $J(n,k,i)$ for $i \in \{0,\dots,k\}$, forms an association scheme, called the Johnson scheme, and denoted by $\mathcal{J}(n,k)$.*

2.4 Types of Intersections

A family $\mathcal{F}$ of subsets of the set $\{1,2,\dots,n\}$ is called *t-intersecting* if any two sets in $\mathcal{F}$ have at least t points in common. If $t=1$, $\mathcal{F}$ is simply called *intersecting* family. In 1961, Erdős, Ko, and Rado proved if $\mathcal{F}$ is a t-intersecting family of k-subsets of $\{1,2,\dots,n\}$, then there is a tight upper bound on the size of $\mathcal{F}$ with n sufficiently large [8].

Theorem 2.4.1 (EKR)**.** *[8] If $\mathcal{F}$ is a t-intersecting family of k-subsets of $\{1,2,\dots,n\}$,* *then there exists a function $f(k,t)$ such that if $n \geq f(k,t)$, then*

$$|\mathcal{F}| \leq \binom{n-t}{k-t}.$$

If equality holds, then $\mathcal{F}$ consists of all k-subsets containing a fixed t-subset of $\{1, 2, \ldots, n\}$.

This theorem was stated in the introduction. As we mentioned in the introduction, the goal of this thesis is to generalize and extend the Erdős-Ko-Rado theorem to uniform set partitions and perfect matchings. In [9], Erdős and Székely considered different types of intersection for partitions. Their work considers all partitions, not just uniform partitions. In [23], Meagher and Moura generalized their definition to uniform set partitions. Here, based on these works, we define three main types of intersections for uniform set partitions.

2.4.1 t-Intersecting Uniform Set Partitions

Two (ℓ, k)-partitions $P = \{P_1, \ldots, P_k\}$ and $Q = \{Q_1, \ldots, Q_k\}$ are said to be *intersecting* if they have at least one block in common. We can generalize this definition to t-intersecting uniform set partitions. For $t < k$, two (ℓ, k)-partitions $P = \{P_1, \ldots, P_k\}$ and $Q = \{Q_1, \ldots, Q_k\}$ are said to be *t-intersecting* if there exist at least t blocks $P_1, \ldots, P_t$ in P and t blocks $Q_1, \ldots, Q_t$ in Q such that $P_i = Q_i$, for all $i = 1, \ldots t$. This is equivalent to $|P \cap Q| \geq t$.

Definition 2.4.2 (t**-Intersecting Uniform Set Partitions**). *A family of (ℓ, k)-partitions $\mathcal{P} \subset \mathcal{U}_{\ell,k}$ is t-intersecting if for all $P, Q \in \mathcal{P}$, $|P \cap Q| \geq t$; in other words*

P and Q are t-intersecting.

In particular, two perfect matchings are said to be t-intersecting if they have at least t edges in common. A family of perfect matchings $\mathcal{P} = \{P_1, \ldots, P_f\}$ in the complete graph K_{2k} are said to be t-intersecting if any pair of perfect matchings $P_i, P_j \in \mathcal{P}$ have at least t edges in common.

The set of all (ℓ, k)-partitions in $\mathcal{U}_{\ell,k}$ that contain a common set of t blocks is called *canonically t-intersecting*. This set is clearly t-intersecting. The size of a set of canonically t-intersecting (ℓ, k)-partitions is

$$\frac{1}{(k-t)!}\binom{k\ell - t\ell}{\ell}\binom{k\ell - t\ell - \ell}{\ell}\binom{k\ell - 2\ell}{\ell}\cdots\binom{\ell}{\ell} = u_{\ell,k-t}. \tag{2.2}$$

For the special case of perfect matchings ($\ell = 2$), the size of a set of canonically t-intersecting perfect matchings of the complete graph on $2k$ vertices is $(2k - 2t - 1)!!$. For a set T of t disjoint edges in K_{2k}, we use ν_T to denote the characteristic vector of the set of all perfect matchings that include all the edges in T. Later, in Chapter 3, we prove an extension of the Erdős-Ko-Rado theorem to 2-intersecting perfect matchings. Further, we prove that if S is a set of 2-intersecting perfect matchings, the characteristic vector of S is a linear combination of the characteristic vectors of the canonically 2-intersecting sets of perfect matchings.

2.4.2 Set-Wise t-Intersecting Uniform Set Partitions

In [6], Ellis used the concept of *set-wise intersecting* families for permutations. A potentially fruitful direction to expand the results in Chapter 3, is to generalize the definition of intersection to set-wise intersection for perfect matchings and uniform set partitions.

Definition 2.4.3 (**Set-Wise t-Intersecting Uniform Set Partitions**). *For $t \leq \lfloor \frac{k}{2} \rfloor$, two (ℓ, k)-partitions P and Q are said to be set-wise t-intersecting if there exist blocks $P_1, \ldots, P_t$ in P and $Q_1, \ldots, Q_t$ in Q such that $\bigcup_{i=1}^{t} P_i = \bigcup_{i=1}^{t} Q_i$. A family of (ℓ, k)-partitions $\mathcal{P}$ is set-wise t-intersecting if any pair of (ℓ, k)-partitions in $\mathcal{P}$ is set-wise t-intersecting.*

Note that in this definition, set-wise t-intersection implies set-wise $(k-t)$-intersection; hence we only consider $t \leq \lfloor \frac{k}{2} \rfloor$. The following example depicts the difference between 2-intersecting and set-wise 2-intersecting perfect matchings in the complete graph K_{2k}. Every pair of t-intersecting perfect matchings is also a pair of set-wise t-intersecting perfect matchings, though the converse is not true.

Example 2.4.4. *In Figure 2.1, each set of coloured edges with the same colour forms a perfect matching in the complete graph K_{2k}. The orange perfect matching and the green perfect matching are both 2-intersecting and set-wise 2-intersecting, while the orange and blue perfect matchings are set-wise 2-intersecting, but not 2-intersecting.*

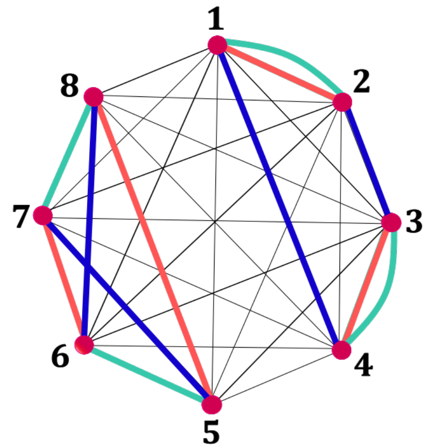

Figure 2.1: 2-intersecting perfect matchings vs set-wise 2-intersecting perfect matchings in K_8

A set of all (ℓ, k)-partitions in $\mathcal{U}_{\ell,k}$ that contains a common set of $t\ell$ elements in exactly t blocks is called *canonically* set-wise t-intersecting and denoted by $\mathcal{S}_t(k, \ell)$. The size of a set of canonically t-intersecting (ℓ, k)-partitions is

$$\frac{1}{(t)!}\binom{t\ell}{\ell}\binom{t\ell-\ell}{\ell}\binom{t\ell-2\ell}{\ell}\cdots\binom{\ell}{\ell}\frac{1}{(k-t)!}\binom{k\ell-t\ell}{\ell}\binom{k\ell-t\ell-\ell}{\ell}\binom{k\ell-2\ell}{\ell}\cdots\binom{\ell}{\ell}, \tag{2.3}$$

which is equal to $u_{\ell,t}u_{\ell,k-t}$. For the special case of perfect matchings ($\ell = 2$), the size of a set of canonically t-intersecting perfect matchings of the complete graph on $2k$ vertices is $(2t-1)!!(2k-2t-1)!!$. It is worthwhile to mention that the set $\mathcal{S}_t(\ell, k)$ is an orbit of the group $\mathrm{Sym}(t\ell) \times \mathrm{Sym}(k\ell - t\ell)$ acting on the partitions.

2.4.3 Partially t-Intersecting Uniform Set Partitions

As we mentioned in Section 2.4, Erdős and Székely considered different types of intersection for partitions [9]. In one of these types, and the one we consider in this section, two partitions P and Q are *intersecting in a pair* if there exist blocks P_i in P, and Q_j in Q such that $|P_i \cap Q_j| \geq 2$. Their work considers all partitions, not just uniform partitions. In [23], Meagher and Moura generalized this definition:

Definition 2.4.5 (**Partially t-intersecting Uniform Set Partitions**). *Two (ℓ, k)-partitions P and Q are partially t-intersecting if there exist P_i in P, and Q_j in Q such that $|P_i \cap Q_j| \geq t$.*

This definition is different from that of Erdős and Székely, since only uniform partitions are considered here. A set of partitions is a *partially t-intersecting set* if any two partitions in the set are partially t-intersecting. Meagher and Moura [23] conjectured that for $t \leq \ell$, if $\mathcal{P} \subset \mathcal{U}_{\ell,k}$ is a set of partially t-intersecting partitions, then $|\mathcal{P}| \leq \binom{k\ell-t}{\ell-t} u_{\ell,k-1}$. A set of this size can be formed by fixing a t-subset T and taking all (ℓ, k)-partitions that have a block containing T; such a set is called a set of *canonically partially t-intersecting (ℓ, k)-partitions*. Moreover, Meagher and Moura conjectured that only the canonically partially t-intersecting (ℓ, k)-partitions have this maximum size. As pointed out by Brunk in [3], this conjecture additionally requires that $\ell \leq k(t-1)$, since if $\ell > k(t-1)$, then any two (ℓ, k)-partitions are t-partially intersecting.

If $\ell = t = 2$, then the $(2, k)$-partitions are perfect matchings in the complete graph on $2k$ vertices. In this case, partially 2-intersecting is equivalent to intersecting (as sets). The Meagher-Moura conjecture has been proven in this case in [16]. In Chapter 5 we will discuss partially 2-intersecting uniform set partitions in more detail.

Chapter 3

t-intersecting Perfect Matchings

The main goal of this chapter is to prove an extension of the Erdős-Ko-Rado theorem to 2-intersecting families of perfect matchings. Specifically, we will prove that for $k \geq 3$ a set of 2-intersecting perfect matchings in K_{2k} of maximum size has $(2k-5)!!$ perfect matchings. The results of this chapter are included in the paper [10].

In the first section of this chapter, we convert the problem of finding the maximum size of an intersecting set of perfect matchings to the problem of finding a maximum coclique in a graph; this graph is a graph in the perfect matching association scheme. In general, finding the largest coclique of a graph is a well-known NP-hard problem, but there is a famous upper bound, called *Delsarte-Hoffman bound* [5], on the size of a maximum coclique. This bound is based on the largest and the smallest eigenvalue for a weighted adjacency matrix. Hence we need to have some information on the eigenvalues of the graph we define in Section 3.1. This leads us to Section 3.2. In Section 3.2 we discuss the character table of the perfect matching association scheme in detail. Using the concept of *quotient graphs*, we construct a portion of the character table of the perfect matching association scheme. Then, in Section 3.3 we provide two different approaches to prove our results. First, we give a proof of the result for some values of k; this proof uses the well-known *clique-coclique* bound (see Subsection 3.3.1). As a second approach, we construct a matrix in the association scheme that is a weighted adjacency matrix for the graph in question. We prove our result by showing the ratio bound holds with equality for this weighted adjacency matrix. In 2012, Tanaka [33] implicitly mentioned how to find such a matrix in P- and Q-polynomial schemes. Tanaka formulated the EKR theorem as a consequence of a linear programming problem, and the feasible solutions are obtained from designs. We will take a similar approach in Section 3.3.2, but we note that, except in the case of $k = 3$, the perfect matching association scheme is neither P- nor Q-polynomial.

3.1 Derangement Graph On Perfect Matchings

We proceed by constructing a graph in which every coclique is a set of intersecting perfect matchings. We then use certain algebraic techniques to find the size of the largest cocliques in this graph.

As we mentioned before, in general, finding the largest coclique of a graph X is a well-known NP-hard problem, but there is a famous upper bound on $\alpha(X)$ that we use throughout this thesis.

Theorem 3.1.1 (Delsarte-Hoffman bound)**.** *[5] Let A be a weighted adjacency matrix for a graph X on vertex set $V(X)$. If A has constant row sums d and least eigenvalue τ, then*

$$\alpha(X) \leq \frac{|V(X)|}{1 - \frac{d}{\tau}}.$$

If equality holds for some coclique S with characteristic vector ν_S, then

$$\nu_S - \frac{|S|}{|V(X)|}\mathbf{1}$$

is an eigenvector with eigenvalue τ.

For a proof of this theorem please see [15, p. 31]. Recall that A weighted adjacency matrix $A_W(X)$ subordinate to X is a symmetric matrix in which rows and columns are indexed by the vertices and the (i, j)-entry may be non-zero (which is interpreted as its edge weight) if $i \sim j$ and is 0 otherwise. Here, our matrices are weighted so that the desired weighted adjacency matrix is in the Bose-Mesner algebra. This bound is based on the ratio between the largest and the smallest eigenvalue for a weighted adjacency matrix, thus it is also known as the *ratio bound*. The ratio bound is important here since we apply it to a graph specifically defined so that the cocliques are precisely sets of 2-intersecting perfect matchings.

Definition 3.1.2. *[16] Define the perfect matching derangement graph $M_t(2k)$ to be the graph whose vertices are perfect matchings on the complete graph K_{2k}. In this graph two vertices are adjacent if they have at most $(t-1)$ edges in common. Denote the adjacency matrix of $M_t(2k)$ by $A_t(2k)$.*

In a coclique of $M_t(2k)$, any two vertices are not adjacent; thus they have more than $t-1$ edges in common or in other words, they are t-intersecting perfect matchings. Using the Delsarte-Hoffman bound, our problem transforms into finding a weighted adjacency matrix for $M_2(2k)$, for any $k \geq 3$, with a sufficiently large ratio between the largest and least eigenvalues. This method to prove EKR theorems was first developed by Wilson in 1984 [35]. In 2015, Godsil and Meagher applied this method to the family of all perfect matchings of the complete graph K_{2k} to find the largest set of intersecting perfect matchings ($t = 1$) [16]; later in 2017 it was applied to t-intersecting perfect matchings by Lindzey [21] for large values of k.

Example 3.1.3 ($M_1(6)$)**.** *In Definition 3.1.2, let $t = 1$ and $2k = 6$. The number of perfect matchings in K_6 is $5!!$, so $M_1(6)$ has 15 vertices. Two vertices here are adjacent if they are not intersecting. Figure 3.1 below depicts $M_1(6)$.*

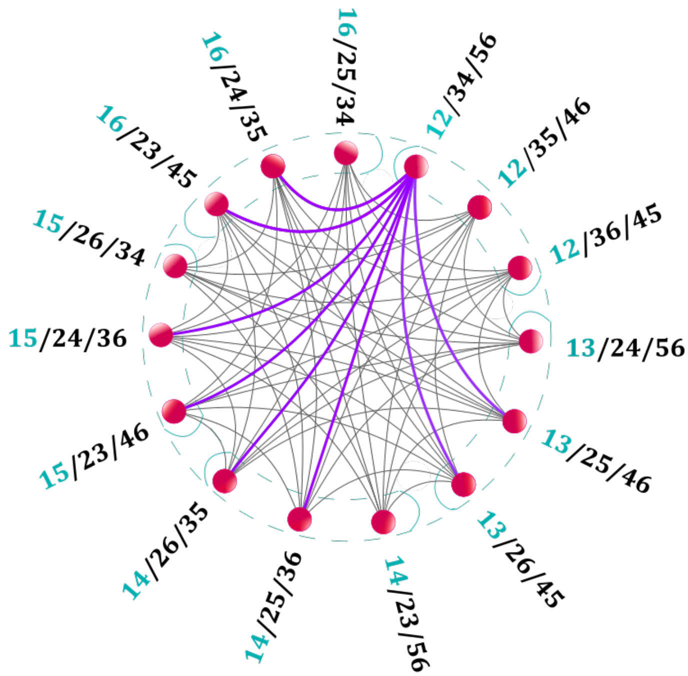

Figure 3.1: Graph $M_t(2k)$ when $t = 1$ and $2k = 6$

The symmetric group $\mathrm{Sym}(2k)$ acts transitively on the set of perfect matchings of K_{2k}. This action preserves the adjacencies in $M_t(2k)$. This implies $\mathrm{Sym}(2k)$ is a subset of the automorphism group of the graph $M_t(2k)$ and that $M_t(2k)$ is vertex transitive. For any λ such that $\lambda \vdash k$, consider the set of all pairs of perfect matchings P and Q, with the property that the union of P and Q has shape 2λ. It is known that $\mathrm{Sym}(2k)$ is transitive on this set of pairs (again see [15, Section 15.4]), this implies each graph $X_{2\lambda}$ defined in Section 2.3 is edge transitive.

Furthermore, the graph $M_t(2k)$ is the union of the graphs of the scheme X_λ in which the even partition λ has at most $t-1$ cycles of length 2, this can be expressed as

$$A_t(2k) = \sum_{\lambda \vdash 2k} A_\lambda,$$

where the sum is taken over partitions λ that have at most $t-1$ parts of length 2. Since the matrices A_λ in the above sum are simultaneously diagonalizable, the eigenvalues of $M_t(2k)$ are the sums of the eigenvalues (from the same eigenspace) of matrices A_λ in the above sum. Thus, in order to infer some information on the eigenvalues of $M_t(2k)$, such as the largest and the smallest eigenvalues, we need to learn more about the character table of the perfect matching association scheme, which is the content of the next section.

3.2 Character Table of the Perfect Matching Association Scheme

In Section 3.3, we find a set of coefficients $\mathbf{a}_{\lambda_\mathbf{i}}$ for the matrices A_{λ_i} in the association scheme of perfect matchings so that the matrix $M = \sum_{i=1}^{m} \mathbf{a}_{\lambda_\mathbf{i}} A_{\lambda_i}$ is a weighted adjacency matrix for the graph $M_2(2k)$ and the ratio bound holds with equality for the matrix M. To verify this, we need to determine the row sums and the least eigenvalue of the matrix M.

If we have the complete character table of the perfect matching association scheme, then we can easily find the row sums and the least eigenvalue of M. The matrices in an association scheme are simultaneously diagonalizable, therefore they have common eigenspaces. This means that the eigenvalue of a linear combination of the matrices A_{λ_i} corresponding to an eigenspace is actually the same linear combination of the eigenvalues of matrices A_{λ_i} corresponding to that eigenspace. Unfortunately finding the complete character table of this scheme for all k is still an unsolved problem. In his 1994 paper, Muzychuk [25] studied the eigenvalues of the association scheme of the symmetric group $\mathrm{Sym}(2k)$. The calculations are quite complicated and Muzychuk only found the eigenvalues up to $2k = 10$. More recently in 2018, Srinivasan [31, 32] presented a recursive algorithm to find the character tables and using this algorithm he was able to calculate the character tables up to $2k = 40$.

In Subsection 3.2.1, we present some background results which are used to calculate the dimension of some of the irreducible $\mathrm{Sym}(n)$-modules in Subsection 3.2.3. Later, in Subsection 3.2.2, we calculate several entries in the character table of the perfect matching association scheme for all values of $2k$. To do this, we calculate the eigenvalues of some carefully chosen quotient graphs. From these eigenvalues we will find an appropriate weighted adjacency matrix.

3.2.1 Kostka Number, and Hook Length Formula

Suppose that $\lambda = [\lambda_1, \dots, \lambda_s]$ is a partition of n. A partition $P = [P_1, \dots, P_s]$, with $|P_i| = \lambda_i$ is called a *tabloid of shape* λ. The group which fixes a tabloid of shape λ is a *Young subgroup* $\mathrm{Sym}(\lambda)$ which is defined as follows,

$$\mathrm{Sym}(\lambda) = \mathrm{Sym}(\lambda_1) \times \mathrm{Sym}(\lambda_2) \times \cdots \times \mathrm{Sym}(\lambda_s).$$

For every integer partition λ of n, there is a permutation representation of $\mathrm{Sym}(n)$ based on its action on the cosets of $\mathrm{Sym}(n)/\mathrm{Sym}(\lambda)$. This is the representation of $\mathrm{Sym}(n)$ induced by the trivial representation on the Young subgroup $\mathrm{Sym}(\lambda)$ [15, Chapter 12]. Denote its character and module by χ_λ and M_λ, respectively. M_λ can be expressed as a sum of irreducible submodules S_μ with $\mu \geq \lambda$ in the dominance ordering. The multiplicity of S_μ in this representation is the *Kostka number* $K_{\mu\lambda}$.

Definition 3.2.1. *[15, Section 12.5] The Young diagram of a partition $\lambda = [\lambda_1, \ldots, \lambda_s]$ is a diagram with n boxes arranged in s left-justified rows, and the i^{th} row contains exactly λ_i boxes.*

In Figure 3.2.1, the Young diagram of the partition $[6, 2, 2]$ has been shown. If μ and λ are two partitions of n, and λ has exactly ℓ parts, then a *semi-standard tableau* with shape μ and content λ is a Young diagram whose boxes are filled with integers $1, \ldots, \ell$ such that, i occupies λ_i boxes, for $i = 1, \ldots, \ell$; and the numbers in each column are strictly increasing, and the numbers in each row are non-decreasing. The Kostka number $K_{\mu,\lambda}$ is the number of semi-standard Young tableaux of shape μ and content λ [15, Section 12.5].

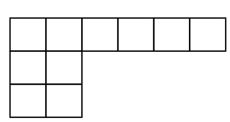

Figure 3.2: Young diagram for the partition $[6, 2, 2]$

Definition 3.2.2. *[15, Section 12.6] The hook length formula for a box in a Young diagram is the number of boxes in the diagram that are either directly below the box or directly to the right of the box, counting the box itself only once.*

The hook length of the box in the ith row from the top, and jth column from the left of a Young diagram is denoted by $h_{i,j}$.

Theorem 3.2.3. *If $\lambda \vdash n$, then the dimension of the irreducible module S_λ is*

$$\frac{n!}{\prod h_{i,j}},$$

where the product is taken over all boxes in the Young diagram for λ.

For two integer partitions $\mu = [\mu_1, \ldots, \mu_\ell]$ and $\lambda = [\lambda_1, \ldots, \lambda_q]$ of n, we say that $\mu \geq \lambda$ in the *dominance ordering* if $\mu_j = \lambda_j$, for all $j < i$ and $\mu_i > \lambda_i$ for some $i \in \{1, \ldots, \min\{q, \ell\}\}$. We will use ϕ_λ to denote the character of $\mathrm{Sym}(n)$ associated to the partition λ.

The decomposition of the representation of $\mathrm{Sym}(n)$ induced by the trivial representation on a Young subgroup is well-known.

Theorem 3.2.4. *[15, Chapter 12] If $\lambda \vdash n$, then*

$$\mathrm{ind}_{\mathrm{Sym}(n)}(1_{\mathrm{Sym}(\lambda)}) = \phi_\lambda + \sum_{\mu > \lambda} K_{\mu\lambda}\phi_\mu,$$

where $K_{\mu\lambda}$ is the Kostka number.

The decomposition of the representation of $\mathrm{Sym}(n)$ induced by the trivial representation on $\mathrm{Sym}(2) \wr \mathrm{Sym}(k)$ is also well-known [15, Chapter 12] and is stated below.

Lemma 3.2.5. *[15, Chapter 12] For integers n and k with $n \geq 2k$,*

$$\mathrm{ind}_{\mathrm{Sym}(n)}(1_{\mathrm{Sym}(2)\wr \mathrm{Sym}(k)}) = \sum_{\lambda \vdash k} \phi_{2\lambda}.$$

3.2.2 Eigenvalues of Perfect Matching Association Scheme

Definition 3.2.6. *[15, Section 2.2] Let $\pi = [\pi_1, \pi_2, \ldots, \pi_i]$ be a set partition of the vertices of the graph X. This partition is equitable if the number of vertices in π_k that are adjacent to a vertex in π_ℓ is determined only by k and ℓ, where $k, \ell \in \{1, \ldots, i\}$. If π is an equitable partition of X, the quotient graph X/π is a directed multi-graph with the parts of π as its vertices, and if a vertex in π_ℓ has exactly ν neighbours in π_k, then X/π has ν arcs from π_ℓ to π_k.*

Quotient graphs are usually represented by a matrix whose rows and columns are indexed by the parts of the equitable partition and whose (π_k, π_ℓ)-entry is the number of edges from a vertex in π_k to π_ℓ.

Consider an integer partition $\lambda = [\lambda_1, \lambda_2, \cdots, \lambda_i]$. The orbit partition formed by the Young subgroup $\mathrm{Sym}(\lambda) = \mathrm{Sym}(\lambda_1) \times \mathrm{Sym}(\lambda_2) \times \cdots \times \mathrm{Sym}(\lambda_i)$ acting on the set of all perfect matchings of K_{2k} (vertices of $M_2(2k)$) is an equitable partition [17, Chapter 9]. Hence for every class μ in the perfect matching association scheme, the quotient graph of X_μ, with respect to this orbit partition, is well-defined. We denote this quotient graph by $X_\mu/\pi(\lambda)$. The eigenvalues of $X_\mu/\pi(\lambda)$ are also eigenvalues of the matrix A_μ [15, p.28]. Using this fact, we will construct certain quotient graphs for several classes in the perfect matching association scheme to build a portion of the character table in the next subsection.

Let A_λ be one of the matrices in the perfect matching association scheme. Let $\mathrm{Sym}(\lambda)$ be a Young subgroup and $A_\lambda/\pi(\lambda)$ represent the corresponding quotient graph. Any eigenvector ν' of the quotient graph can be *lifted* to form an eigenvector ν for A_λ (the P-entry of ν is equal to the entry of ν' corresponding to the part that contains P). The groups $\mathrm{Sym}(\lambda)$ and $\mathrm{Sym}(n)$ both act on the cosets of $\mathrm{Sym}(n)/\left(\mathrm{Sym}(2) \wr \mathrm{Sym}(k)\right)$, and thus also act on the vector ν by permuting the indices. Since the entries of ν are constant on the orbits of $\mathrm{Sym}(\lambda)$, the vector ν is unchanged by the action of $\mathrm{Sym}(\lambda)$. Define

$$V = \mathrm{span}\{\sigma\nu : \sigma \in \mathrm{Sym}(n)\}.$$

In particular, V is the $\mathrm{Sym}(n)$-module generated by the action of $\mathrm{Sym}(n)$ on ν.

Our plan is to use the Young subgroup to form the quotient graphs. The eigenvalues of the quotient graphs belong to the modules that are in both decompositions in Theorem 3.2.4 and Lemma 3.2.5.

Theorem 3.2.7. *Assume that* $\mathrm{Sym}(n)$ *acts on the set* Ω*, and that* A *is the adjacency matrix for an orbital of the action of* $\mathrm{Sym}(n)$ *on* Ω*. Let* $\lambda \vdash n$ *and* π *be the orbit partition from the action of* $\mathrm{Sym}(\lambda)$ *on* Ω*. If* η *is an eigenvalue of the quotient graph* A/π*, then* η *is an eigenvalue of* A*. Moreover,* η *belongs to some* $\mathrm{Sym}(n)$*-module represented by the partition* μ *where* $\mu \geq \lambda$ *in the dominance ordering.*

Proof. Let π be the orbit partition of $\mathrm{Sym}(\lambda)$ acting on Ω. Assume that ν' is an eigenvector of the quotient graph A/π with eigenvalue η. The vector ν' can be lifted to an η-eigenvector of A, which we denote by ν.

The group $\mathrm{Sym}(n)$ and $\mathrm{Sym}(\lambda)$ both act on Ω and thus also act on the vector ν by permuting the entries. Since the entries of ν are constant on the orbits of $\mathrm{Sym}(\lambda)$, the vector ν is unchanged by the action of $\mathrm{Sym}(\lambda)$. Define a vector space W to be the span of the vector ν, so $\mathrm{Sym}(\lambda)$ fixes every element in W.

Set $V = \oplus_{\sigma \in \mathrm{Sym}(n)} \sigma W$. Then V is isomorphic to the module for the induced representation $\mathrm{ind}_{\mathrm{Sym}(n)}(1_{\mathrm{Sym}(\lambda)}) = \phi_\lambda + \sum_{\mu > \lambda} K_{\mu\lambda} \phi_\mu$. Clearly the vector $\nu \in V$ and since ν is an η-eigenvector there is a $\mu \geq \lambda$ so that the μ-module is a subspace of the η-eigenspace. □

Example 3.2.8. *Consider the matrix* $A_{[2k-4,4]}$ *in the perfect matching association* *scheme. Note that in the graph corresponding to this matrix, two perfect matchings are adjacent if and only if their union forms a* 4*-cycle and a* $(2k-4)$*-cycle. The following matrix is the quotient graph corresponding to the group* $\mathrm{Sym}(2k-2)\times\mathrm{Sym}(2)$*. Denote this quotient graph by* $A(X_{[2k-4,4]})/\pi([2k-2,2])$ *(note that in this notation the first integer partition is the class in the perfect matching association scheme, the second partition is the Young subgroup used to form the partition of the vertices in the graph). The group* $\mathrm{Sym}(2k-2)\times\mathrm{Sym}(2)$ *has two orbits on perfect matchings of size* $2k$*:* S_1 *the set of all perfect matchings that have 1 and 2 in the same edge, and* S_2 *the set of all perfect matchings in which 1 and 2 are in different edges. The rows and columns of the matrix* $A(X_{[2k-4,4]})/\pi([2k-2,2])$ *are indexed by these two orbits, and for any perfect matching* P *in* S_i*, the* (i,j)*-entry counts the number of perfect matchings in* S_j *that are adjacent to* P *(since the partition of perfect matchings to* S_1*, and* S_2 *is equitable, we get the same number for any choice of* P *in* S_1*). The* $(1,1)$*-entry is* 0 *because* S_1 *is a coclique. As the row sum in* $A(X_{[2k-4,4]})/\pi([2k-2,2])$ *is equal to the degree, the* $(1,2)$*-entry is the degree. Without loss of generality, consider the perfect matching* $Q:\{1,3\},\{2,4\},\{5,6\},\ldots,\{2k-1,2k\}$ *in* S_2*. There are two types of perfect matchings in* S_1 *that are adjacent to* Q*. First, the perfect matchings which contain edges* $\{1,2\},\{3,4\}$*. The number of such perfect matchings that are adjacent to* Q *is* $(2k-6)!!$*. Second, consider the perfect matchings in* S_1 *in which* $\{1,2\}$ *is an edge but 3 and 4 are in separate edges. The number of such perfect matchings* *that are adjacent to* Q *is* $2\binom{k-2}{2}(2k-8)!! = \frac{1}{2}(k-2)(2k-6)!!$. *So the* $(2,1)$*-entry is* $\frac{1}{2}k(2k-6)!!$. *Similarly we can calculate the* $(2,2)$*-entry and the following is the matrix* $A(X_{[2k-4,4]})/\pi([2k-2,2])$.

$$A(X_{[2k-4,4]})/[2k-2,2] = \left[\begin{array}{c|c} \mathbf{0} & k(k-1)(2k-6)!! \\ \hline \frac{1}{2}k(2k-6)!! & \frac{1}{2}k(2k-3)(2k-6)!! \end{array}\right].$$

For the matrix $A(X_{[2k-4,4]})/\pi([2k-2,2])$ *the all-ones vector* $\mathbf{1}$ *is an eigenvector corresponding to the largest eigenvalue,* $k(k-1)(2k-6)!!$. *This eigenvalue is actually the degree of* $A_{[2k-4,4]}$, *and by Theorem 3.2.7, this eigenvalue corresponds to the* $[2k]$*-module in the character table. It is well-known that the trace of a matrix is equal to the sum of its eigenvalues, so by subtracting the degree from the trace, we find the second eigenvalue of this matrix which is* $-\frac{1}{2}k(2k-6)!!$. *Using Theorem 3.2.7, and noting that the degree eigenvalue belongs to the* $[2k]$*-module, it is easy to deduce that the second eigenvalue belongs to the* $[2k-2,2]$*-module.*

The quotient matrix of $A_{[2k-4,4]}$ *with the orbit partition formed by the action of the* *group* $\mathrm{Sym}(2k-4) \times \mathrm{Sym}(4)$ *on the perfect matchings is the following*

$$\left[\begin{array}{c|c|c} 2(2k-6)!! & (2k-4)!! & (k-1)(k-2)(2k-6)!! \\ \hline \frac{1}{2}(2k-6)!! & \frac{1}{2}(5k-2)(2k-6)!! & \frac{1}{2}(2k^2-7k+1)(2k-6)!! \\ \hline \frac{3}{2}(k-1)(2k-8)!! & 3(2k^2-7k+1)(2k-8)!! & (2k^3-14k^2+\frac{51}{2}k-\frac{3}{2})(2k-8)!! \end{array}\right].$$

This matrix is denoted by $A(X_{[2k-4,4]})/\pi([2k-4,4])$*. This matrix yields the eigenvalues of* $A_{[2k-4,4]}$ *belonging to the modules* $[2k]$*,* $[2k-2,2]$*, and* $[2k-4,4]$*. From the matrix* $A(X_{[2k-4,4]})/\pi([2k-2,2])$ *we already have two eigenvalues of the matrix* $A(X_{[2k-4,4]})/\pi([2k-4,4])$*, those corresponding to the modules* $[2k]$ *and* $[2k-2,2]$*. Hence by subtracting these two eigenvalues from the trace, we find a third eigenvalue of* $A_{[2k-4,4]}$*, the one corresponding to the module* $[2k-4,4]$*, and it is equal to* $\frac{1}{2}(7k-15)(2k-8)!!$.

By finding several quotient graphs for the classes $[2k]$, $[2k-2,2]$, $[2k-4,4]$, $[2k-4,2,2]$, and $[2k-6,6]$ in the perfect matching association scheme, we construct a portion of the character table for the association scheme on the perfect matchings for K_{2k} for any $k \geq 6$. These calculations were difficult but rewarding. The counting arguments are similar to the one in Example 3.2.8. By using Theorem 3.2.7 to define the quotient graphs of several classes in this association scheme, we can use the dominance ordering recursively to determine the eigenvalues that belong to some of the modules in the character table. We present the results as the following theorems.

Theorem 3.2.9. *Let $k \geq 6$. For the class $[2k]$ in the perfect matching association scheme, the eigenvalues corresponding to the modules $[2k]$, $[2k-2,2]$, $[2k-4,4]$, $[2k-4,2,2]$, and $[2k-6,6]$, are $\frac{(2k)!!}{2k}$, $-(2k-4)!!$, $-(2k-6)!!$, $2(2k-6)!!$, and $-3(2k-8)!!$, respectively.*

Theorem 3.2.10. *Let $k \geq 6$. For the class $[2k-2,2]$ in the perfect matching association scheme, the eigenvalues corresponding to the modules $[2k]$, $[2k-2,2]$, $[2k-4,4]$, $[2k-4,2,2]$, and $[2k-6,6]$, are $\frac{(2k)!!}{2(2k-2)}$, $\frac{(2k-4)!!}{2}$, $-(5k-12)(2k-8)!!$, $-(2k-6)!!$, and $-3(3k-10)(2k-10)!!$, respectively.*

Theorem 3.2.11. *Let $k \geq 6$. For the class $[2k-4,4]$ in the perfect matching association scheme, the eigenvalues corresponding to the modules $[2k]$, $[2k-2,2]$, $[2k-4,4]$, $[2k-4,2,2]$, and $[2k-6,6]$, are $\frac{(2k)!!}{4(2k-4)}$, $\frac{-2k(2k-6)!!}{4}$, $\frac{(7k-15)(2k-8)!!}{2}$, $\frac{-(2k-6)!!}{2}$, and $-3(9k^2-71k+140)(2k-12)!!$, respectively.*

Theorem 3.2.12. *Let $k \geq 6$. For the class $[2k-4,2,2]$ in the perfect matching association scheme, the eigenvalues corresponding to the modules $[2k]$, $[2k-2,2]$, $[2k-4,4]$, $[2k-4,2,2]$, and $[2k-6,6]$, are $\frac{(2k)!!}{8(2k-4)}$, $\frac{(3k-2)(2k-6)!!}{4}$, $\frac{-(k+3)(2k-8)!!}{4}$, $(k^2-7k+12)(2k-10)!!$, and $\frac{-3}{2}(13k^2-101k+190)(2k-12)!!$, respectively.*

Theorem 3.2.13. *Let $k \geq 6$. For the class $[2k-6,6]$ in the perfect matching association scheme, the eigenvalues corresponding to the modules $[2k]$, $[2k-2,2]$,*

$[2k-4,4]$, $[2k-4,2,2]$, *and* $[2k-6,6]$, *are* $\frac{(2k)!!}{6(2k-6)}$, $\frac{-2k(2k-4)!!}{6(2k-6)}$, $\frac{-2k(2k-6)!!}{6(2k-6)}$, $\frac{4k(2k-6)!!}{6(2k-6)}$, *and* $6(5k^2-38k+70)(2k-12)!!$, *respectively.*

These results of the Theorems 3.2.9, 3.2.10, 3.2.11, 3.2.12, and 3.2.13 are recorded and summarized in Table 3.2.2.

	$[2k]$	$[2k-2,2]$	$[2k-4,4]$	$[2k-4,2,2]$	$[2k-6,6]$
$\chi_{[2k]}$	$\frac{(2k)!!}{2k}$	$\frac{(2k)!!}{2(2k-2)}$	$\frac{(2k)!!}{4(2k-4)}$	$\frac{(2k)!!}{8(2k-4)}$	$\frac{(2k)!!}{6(2k-6)}$
$\chi_{[2k-2,2]}$	$-(2k-4)!!$	$\frac{(2k-4)!!}{2}$	$\frac{-2k(2k-6)!!}{4}$	$\frac{(3k-2)(2k-6)!!}{4}$	$\frac{-2k(2k-4)!!}{6(2k-6)}$
$\chi_{[2k-4,4]}$	$-(2k-6)!!$	$-(5k-12)(2k-8)!!$	$\frac{(7k-15)(2k-8)!!}{2}$	$\frac{-(k+3)(2k-8)!!}{4}$	$\frac{-2k(2k-6)!!}{6(2k-6)}$
$\chi_{[2k-4,2,2]}$	$2(2k-6)!!$	$-(2k-6)!!$	$\frac{-(2k-6)!!}{2}$	$(k^2-7k+12)(2k-10)!!$	$\frac{4k(2k-6)!!}{6(2k-6)}$
$\chi_{[2k-6,6]}$	$-3(2k-8)!!$	$-3(3k-10)(2k-10)!!$	$-3(9k^2-71k+140)(2k-12)!!$	$\frac{-3}{2}(13k^2-101k+190)(2k-12)!!$	$6(5k^2-38k+70)(2k-12)!!$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

We prove another result beyond Theorem 3.2.7 that will be used later in this chapter. For a set $S \subset \{1, \ldots, 2k\}$ of size four, consider the set of all perfect matchings that include two edges e and e' such that e and e' form a set partition of S. This means that S is contained in only 2 edges. This set is the first part in the equitable partition formed by the action of $\mathrm{Sym}([2k-4, 4])$ on the perfect matchings given in Example 3.2.8. For example, if $S = \{1, 2, 3, 4\}$ the characteristic vector of this set is

$$w_{\{1,2,3,4\}} = \nu_{\{1,2\},\{3,4\}} + \nu_{\{1,3\},\{2,4\}} + \nu_{\{1,4\},\{2,3\}}.$$

Here, $\nu_{\{1,2\},\{3,4\}}$ is a $(0, 1)$-vector indexed by perfect matchings, and it has entry 1 if and only if the corresponding perfect matching has edges $\{1, 2\}, \{3, 4\}$. Similarly we can define vectors $\nu_{\{1,3\},\{2,4\}}$ and $\nu_{\{1,4\},\{2,3\}}$. This means that w_S denotes the characteristic vector for the set of perfect matchings in which an arbitrary 4-set S is contained in only 2 edges.

Theorem 3.2.14. *The set* $\{w_S \,|\, S \subset \{1, 2, \ldots, 2k\}$ *with* $|S| = 4\}$ *is a spanning set for the* $\mathrm{Sym}(2k)$ *module* $\mathrm{span}\{[2k], [2k-2, 2], [2k-4, 4]\}$*, where* $[2k]$*,* $[2k-2, 2]$*, and* $[2k-4, 4]$ *represent the modules.*

Proof. Consider the quotient matrix $A(X_{[2k-4,4]})/\pi([2k-4, 4])$ formed by the action of $\mathrm{Sym}([2k-4, 4])$ on the perfect matchings (this matrix is given in Example 3.2.8). The first part in the equitable partition is the set of all perfect matchings for which a fixed set of size S four is contained in only two edges (that is, S is the 4-set stabilized by $\mathrm{Sym}([2k-4, 4])$).

The matrix $A(X_{[2k-4,4]})/\pi([2k-4,4])$ is a 3×3-matrix that is diagonalizable. This implies that the vector $(1,0,0)$ can be expressed as a linear combination of the eigenvectors of the quotient matrix. Further, each of the eigenvectors of the quotient matrix can be lifted to be an eigenvector for the adjacency matrix. By Theorem 3.2.7, these lifted vectors are the eigenvectors belonging to the $[2k]$, $[2k-2,2]$ and $[2k-4,4]$ modules of $\mathrm{Sym}(2k)$.

Using the same linear combination to produce the vector $(1,0,0)$ from the eigenvectors of the quotient matrix, w_S is a linear combination of eigenvectors for the modules $[2k]$, $[2k-2,2]$ and $[2k-4,4]$ (indeed w_S is the vector formed by lifting $(1,0,0)$). So we conclude for any subset S of size four, the vector w_S is in the span of the $[2k]$, $[2k-2,2]$ and $[2k-4,4]$ modules.

Finally, we show that the dimension of the span of w_S where S is taken over all 4-subsets of $\{1,2,\ldots,2k\}$ is equal to the dimension of the span of the $[2k]$, $[2k-2,2]$ and $[2k-4,4]$ modules. Define N to be the matrix with the rows indexed by the perfect matchings of K_{2k} and the columns by the 4-subsets $S \subset \{1,2,\ldots,2k\}$. Columns of N are the vectors w_S. The entries of N^TN depend only on the size of the intersection of the 4-subsets, so it can be written as a linear combination of the matrices in the Johnson scheme $\mathcal{J}(2k,4)$. In particular

$$N^TN = (2k-5)!!I + (2k-7)!!J(2k,4,2) + 9(2k-9)!!J(2k,4,4).$$

Here, $J(2k,4,i)$ is the adjacency matrix of the graph in which vertices are the 4-subsets of a set of size $2k$; and two vertices are adjacent if their intersection is of size $4-i$. The eigenvalues of the Johnson scheme are well-known and can be used to calculate the eigenvalues of N^TN. It is straight-forward to see that 0 is an eigenvalue of N^TN with multiplicity $2k-1+\binom{2k}{4}-\binom{2k}{3}$. This implies the rank of N^TN, and hence the rank of N, equals the dimension of $[2k]$, $[2k-2,2]$ and $[2k-4,4]$ modules. Thus the set $\{w_S \mid S \subset \{1,\ldots,2k\} \text{ with } |S|=4\}$ is a spanning set. □

3.2.3 The Degrees of the Irreducible Modules of Sym(n)

In this subsection we review some results on the dimension of the irreducible modules of Sym($2k$). Later we use these results to prove Theorem 3.3.7.

Let $\lambda = [\lambda_1, \lambda_2, \cdots, \lambda_t]$ be an integer partition of $2k$; the dimension of the λ module will be denoted by $m(\lambda)$. The *dual partition* λ^* to the partition λ is the partition with the Young diagram that is the reflection of the Young diagram of λ. The degree of a partition and its dual is the same; denoted by $m(\lambda) = m(\lambda^*)$. A partition λ is called *primary* if $\lambda \geq \lambda^*$ in the dominance ordering (see [27] for more details).

Theorem 3.2.15. *[27, p.151] Let $\lambda = [\lambda_1, \lambda_2, \cdots, \lambda_t]$ be an integer partition of $2k$ in which $\lambda_1 \geq k$. Then,*

$$m([\lambda_1, 2k-\lambda_1]) \leq m(\lambda).$$

The next result follows from a straight-forward application of the hook length formula.

Lemma 3.2.16. *[15, Section 12.6] Let $n \geq 2s$, then*

$$m([n-s,s]) = \binom{n}{s} - \binom{n}{s-1}.$$

The next result is a general bound on the degree of a representation in which the first part of the corresponding integer partition is considered small.

Theorem 3.2.17. *[27, p.163] Let λ be a primary partition of n for which the first part $\lambda_1 < \lfloor \frac{n}{2} \rfloor$. Then $m(\lambda) \geq F(n)$, where*

$$F(n) = \begin{cases} n \cdot F(n-1)(m+2) & \text{if } n = 2m+1 \text{ is odd,} \\ 2 \cdot F(n-1) & \text{if } n \text{ is even,} \end{cases}$$

with $F(0) = 2$. In particular, for $n \geq 8$,

$$\frac{3}{2} \cdot F(n-1) \leq F(n) \leq 2 \cdot F(n-1). \tag{3.4}$$

3.3 EKR Theorem for 2-intersecting Perfect Matchings

In this section, we provide two different approaches to prove the Erdős-Ko-Rado theorem for 2-intersecting families of perfect matchings of the complete graph K_{2k}.

3.3.1 Clique-Coclique Approach

For our first approach, we construct a large clique in $M_2(2k)$ and then apply the clique-coclique bound. This clique is constructed using *projective planes* which are introduced and discussed in the next subsection. This approach only works for some values of k. The existence of such a clique implies the existence of a weighted adjacency matrix M of $M_2(2k)$, for which the Delsarte-Hoffman bound holds with equality 3.3.5. This is similar to the result in [13], where the authors show how to find the matrix used in Wilson's proof of the EKR Theorem [35]. This is the motivation of our second approach, where we show that such a matrix exists, even when there is no appropriate clique.

Definition 3.3.1. *[34, p.51] A finite projective plane $PG(2, n)$ consists of a finite set of points P of size $n^2 + n + 1$ and a set L of subsets of P, called lines, satisfying the axioms (P1), (P2), and (P3):*

(P1) Given two points in P, there is exactly one line that contains both.

(P2) Given two lines in L, there is exactly one point on both.

(P3) There are four points of which no three are co-linear.

As a result of the above definition, every line contains $n + 1$ points. Assume $\mathcal{O}$ is a set of points in a $PG(2, n)$. A line l is called a *secant* of $\mathcal{O}$ if l and $\mathcal{O}$ have 2 points in common; if they have 1 point in common, l is called a *tangent* of $\mathcal{O}$. If any line has at most 2 points in common with $\mathcal{O}$, then $\mathcal{O}$ is called a 2-*arc*. A 2-arc with $n + 1$ points is called an *oval*, and a 2-arc with $n + 2$ points is called a *hyperoval.*

Theorem 3.3.2. *[34, p.56] For every positive integer a, there is a $PG(2, 2^a)$ containing a hyperoval.*

We know that $X_2(2k)$ is a vertex-transitive graph. We apply the following well-known bound on the size of a coclique on a vertex-transitive graph.

Theorem 3.3.3 (Clique-Coclique Bound)**.** *[15, p.26] Let X be a vertex-transitive graph. Then*

$$\alpha(X)\,\omega(X) \leq |V(X)|.$$

If equality holds for a clique C and a coclique S, then the vectors

$$\chi_C - \frac{|C|}{v}\mathbf{1}, \quad \chi_S - \frac{|S|}{v}\mathbf{1}$$

are orthogonal, where χ_C and χ_S are the characteristic vectors of C and S respectively. In particular $\chi_C^T \chi_S = 1$.

In the next theorem we show that equality in the clique-coclique bound holds for $M_2(2k)$ for infinitely many values of k.

Theorem 3.3.4. *If $2k = 2^a + 2$, where a is a positive integer, then $\omega(M_2(2k)) = (2k-1)(2k-3)$.*

Proof. By Theorem 3.3.2, for every integer a there exists a $PG(2, 2^a)$ containing a hyperoval. Call this hyperoval $\mathcal{O}$. Clearly $|\mathcal{O}| = 2k = 2^a + 2$. Let $\mathcal{S}$ be the set of points in $\mathcal{O}$. Note that $|\mathcal{S}| = 2^{2a} - 1$. For each $s \in \mathcal{S}$, let $\mathcal{L}_s$ be the set of lines of secants through s. Clearly, each point of $\mathcal{O}$ is covered by exactly one element of $\mathcal{L}_s$, and so $|\mathcal{L}_s| = \frac{\mathcal{O}}{2} = k$.

We can identify the points of $\mathcal{O}$ with the vertices of K_{2k}. Every line in $\mathcal{L}_s$ forms an edge of K_{2k} as it has exactly 2 points in common with $\mathcal{O}$, and as the elements of $\mathcal{L}_s$ only meet in s, $\mathcal{L}_s$ corresponds to a perfect matching of K_{2k}. Thus, the set $\mathcal{C} = \{\mathcal{L}_s | s \in \mathcal{S}\}$ forms a family of perfect matchings of size $2^{2a}-1 = (2^a+1)(2^a-1) = (2k-1)(2k-3)$. As two points lie on exactly one line together, $|\mathcal{L}_s \cap \mathcal{L}_r| \leq 1$ for $s, r \in \mathcal{S}$, $s \neq r$. Hence, $\mathcal{C}$ is actually a clique. Finally we note that $\mathcal{C}$ is a maximum clique in M_{2k} since a canonical 2-intersecting perfect matching is a coclique of size $(2k-5)!!$; hence equality in the clique-coclique bound holds.

$\square$

In the following theorem, we use the concept of *positive semi-definite matrices*. A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is positive semi-definite (psd) if $v^t A v \geq 0$, for all $v \in \mathbb{R}^n$. Equivalently, a symmetric matrix is psd if and only if all its eigenvalues are non-negative.

Theorem 3.3.5. *Suppose that X is a graph in a symmetric association scheme. If there is a clique in X for which equality holds in the clique-coclique bound, then there exists a weighted adjacency matrix of X, for which the ratio bound holds with equality.*

Proof. Let $\mathcal{A} = \{A_0, A_1, \ldots, A_k\}$ be a symmetric association scheme on v vertices and denote the row sum of A_i by v_i. Let X_i be the graph associated with A_i, for $i \in \{1, \ldots, k\}$. Let X be a graph such that $X = \bigcup_{i \in T} X_i$ for some $T \subset \{1, \ldots, k\}$.

Let C be a clique in X for which the clique-coclique bound holds with equality, so $\alpha(X) = v/|C|$. Let χ_C be the characteristic vector of C. Then $\chi_C^T \chi_C$ is a positive semi-definite (psd) matrix. The projection of this matrix into the Bose-Mesner algebra is the matrix

$$\widehat{M} = \sum_{i=0}^{k} \frac{\chi_C^T A_i \chi_C}{v v_i} A_i.$$

Clearly, $\widehat{M}$ is again a psd matrix (see [13, Lemma 3.2]). Since $A_0 = I$, we have $\frac{\chi_C^T A_0 \chi_C}{v v_0} A_0 = \frac{|C|}{v} I$.

Define $M = \widehat{M} - \frac{|C|}{v} I$. Since $\widehat{M}$ is psd, the minimal eigenvalue for M is at least $-\frac{|C|}{v}$.

Further, since C is a clique in X, any two vertices in C will be related in some A_i for $i \in T$, in particular $\chi_C A_j \chi_C = 0$ for $j \notin T$. Thus

$$M = \sum_{i=1}^{k} \frac{\chi_C^T A_i \chi_C}{v v_i} A_i = \sum_{i \in T} \frac{\chi_C^T A_i \chi_C}{v v_i} A_i$$

and M is a weighted adjacency matrix of X.

The row sums of $\widehat{M}$ can be calculated as follows

$$\widehat{M}\mathbf{1} = \sum_{i=0}^{k} \frac{\chi_C^T A_i \chi_C}{v v_i} A_i \mathbf{1} = \frac{1}{v} \sum_{i=0}^{k} \chi_C^T A_i \chi_C \mathbf{1} = \frac{1}{v} \chi_C^T \left(\sum_{i=0}^{k} A_i \right) \chi_C \mathbf{1} = \frac{1}{v} \chi_C^T J \chi_C \mathbf{1} = \frac{|C|^2}{v} \mathbf{1}.$$

Hence the row sums of M all equal

$$d = \frac{|C|^2}{v} - \frac{|C|}{v}.$$

We also know the minimum eigenvalue τ is negative and at least $-\frac{|C|}{v}$ and the maximum coclique has size $\frac{v}{|C|}$. The ratio bound, applied to M gives

$$\frac{v}{|C|} = \alpha(X) \leq \frac{v}{1-\frac{d}{\tau}} = \frac{v}{1-\frac{\frac{|C|^2}{v}-\frac{|C|}{v}}{\tau}}.$$

Rearranging verifies that τ is less than or equal to $-\frac{|C|}{v}$. Thus $\tau = -\frac{|C|}{v}$ and equality holds in the ratio bound. □

We have proved that for infinitely many values of $2k = 2^a+2$ we can build a maximum clique in $M_2(2k)$, hence we can construct the matrix M as in Theorem 3.3.5 with equality in the ratio bound. The value $2^a + 2$ grows fast as a increases, which makes it difficult to find the maximum clique, and the values in M by computer. In the next approach we will build the matrix M without identifying maximum cliques.

3.3.2 Least Eigenvalue Trick

In this subsection, our goal is to show that the set of perfect matchings with two fixed edges is a maximum coclique in $M_2(2k)$. To address this, we determine an appropriate set of coefficients $a_{2\lambda}$ so that

$$M = \sum_{\lambda \vdash k} a_{2\lambda} A_{2\lambda} \tag{3.5}$$

is a weighted adjacency matrix of $M_2(2k)$ with row sum $(2k-1)(2k-3)-1$ and the least eigenvalue -1. This proves that the ratio bound

$$\alpha(X) = \frac{|V(X)|}{1-\frac{d}{\tau}} = \frac{(2k-1)!!}{1-\frac{(2k-1)(2k-3)-1}{-1}} = (2k-5)!!$$

holds with equality for $M_2(2k)$. To be a weighted adjacency matrix of $M_2(2k)$, we need that $a_{2\lambda} = 0$ whenever λ has 2 or more ones. Further, the eigenvalue of M corresponding to the μ-module is $\xi^\mu = \sum_{\lambda \vdash k} a_{2\lambda} \xi^\mu_{2\lambda}$, where $\xi^\mu_{2\lambda}$ is the eigenvalue of $A_{2\lambda}$ belonging to the μ-module.

Theorem 3.3.6. *For $3 \leq k \leq 9$, there exists a weighted adjacency matrix of the graph $M_2(2k)$ for which the degree and the least eigenvalue are $(2k-1)(2k-3)-1$ and -1, respectively.*

Proof. For $k = 3, 4, 5$, define the matrices $M_6 = A_{[6]} + A_{[4,2]}$, $M_8 = \frac{1}{4}A_{[8]} + \frac{1}{2}A_{[6,2]} + \frac{1}{2}A_{[4,4]}$, and $M_{10} = \frac{1}{12}A_{[10]} + \frac{1}{12}A_{[8,2]} + \frac{1}{6}A_{[4,4,2]}$. The matrices M_6, M_8, and M_{10} are the desired weighted adjacency matrices for the graphs $M_2(6)$, $M_2(8)$, and $M_2(10)$, respectively.

	$A_{[8]}$	$A_{[6,2]}$	$A_{[4,4]}$	$A_{[4,2,2]}$	$A_{[2,2,2,2]}$	M_8
$\chi_{[8]}$	48	32	12	12	1	34
$\chi_{[6,2]}$	-8	4	-2	5	1	-1
$\chi_{[4,4]}$	-2	-8	7	2	1	-1
$\chi_{[4,2,2]}$	4	-2	-2	-1	1	-1
$\chi_{[2,2,2,2]}$	-6	8	3	-6	1	4

Table 3.1: Character table for $2k = 8$

As we see in Table 3.1, the row sum and the least eigenvalue of M_8 are $(2k-1)(2k-$

3) − 1 and −1. Similarly, using the character tables for $k = 3$ and $k = 5$ [25, 31], we find the eigenvalues of the matrices M_6 and M_{10} and verify that the ratio bound holds with equality.

For $k = 6$, Theorem 3.3.4 proves that equality holds in the ratio bound. For $7 \leq k \leq 9$, we have the complete character table for the perfect matching association scheme [32]. So we can express the eigenvalues of $M = \sum_{\lambda \vdash k} a_{2\lambda} A_{2\lambda}$ as a system of linear equations. The objective is to maximize the value of the greatest eigenvalue (this is the row sum, so the eigenvalue belonging to the $[2k]$ module) while fixing the eigenvalues corresponding to the modules $[2k-2, 2]$, $[2k-4, 4]$, and $[2k-4, 2, 2]$ to be -1, and having all other eigenvalues strictly greater than -1. This will be explained later. The Gurobi Optimizer [18] is then used to find solutions for these systems of inequalities. As such we determined the desired weighted adjacency matrices as follows:

$$
\begin{aligned}
M_7 &= \frac{1}{640} A_{[14]} + \frac{1}{80} A_{[6,6,2]} + \frac{1}{60} A_{[4,4,4,2]}, \\
M_8 &= \frac{1}{3840} A_{[14,2]} + \frac{1}{2048} A_{[10,6]} + \frac{1}{120} A_{[4,4,4,4]}, \\
M_9 &= \frac{1}{80640} A_{[18]} + \frac{1}{13440} A_{[8,8,2]} + \frac{1}{4480} A_{[6,6,4,2]}.
\end{aligned}
$$

□

To find the set of coefficients for $k \geq 10$, we consider linear combinations of the form

$$
M_k = \mathbf{a_1} A_{[2k]} + \mathbf{a_2} A_{[2k-2,2]} + \mathbf{a_3} A_{[2k-4,4]}.
$$

To find the values of $\mathbf{a_1}, \mathbf{a_2}$ and $\mathbf{a_3}$ for $k \geq 10$, we use the eigenvalues in the partial character table in Table 3.2.2 to produce a corresponding linear system. For this system, there is one equation for each of the eigenvalues that correspond to the irreducible modules $[2k-2,2]$, $[2k-4,4]$, and $[2k-4,2,2]$, which are equated to -1. The rationale for choosing these modules is that they, along with $[2k]$, are the modules that are in both the decomposition of $\mathrm{ind}_{\mathrm{Sym}(n)}(1_{\mathrm{Sym}([2k-4,2,2])})$ and $\mathrm{ind}_{\mathrm{Sym}(n)}(1_{\mathrm{Sym}(2)\wr\mathrm{Sym}(k)})$. Observe $\mathrm{Sym}([2k-4,2,2])$ is the group that stabilizes the set of all perfect matchings which include a fixed pair of edges.

Using the results in Subsection 3.2.2, this linear system becomes:

$$\begin{aligned}
-(2k-4)!!\mathbf{a_1} + \quad (k-2)(2k-6)!!\mathbf{a_2} - \quad k(k-3)(2k-8)!!\mathbf{a_3} &= -1,\\
-(2k-6)!!\mathbf{a_1} - (5k-12)(2k-8)!!\mathbf{a_2} + \tfrac{1}{2}(7k-15)(2k-8)!!\mathbf{a_3} &= -1,\\
2(2k-6)!!\mathbf{a_1} - \qquad (2k-6)!!\mathbf{a_2} - \qquad \tfrac{1}{2}(2k-6)!!\mathbf{a_3} &= -1.
\end{aligned}$$

Solving this system, we obtain the coefficients $\mathbf{a_1} = \frac{1}{4(2k-6)!!}$, and $\mathbf{a_2} = \mathbf{a_3} = \frac{1}{(2k-6)!!}$. Note that for $k > 4$, the determinant of the coefficient matrix corresponding to the aforementioned linear system is nonzero, so the values $\mathbf{a_1}$, $\mathbf{a_2}$, and $\mathbf{a_3}$ are unique. This can be easily checked by using any basic mathematical software.

Theorem 3.3.7. *For $k \geq 10$, let $M = \mathbf{a_1}A_{[2k]} + \mathbf{a_2}A_{[2k-2,2]} + \mathbf{a_3}A_{[2k-4,4]}$ where $\mathbf{a_1} = \frac{1}{4(2k-6)!!}$, and $\mathbf{a_2} = \mathbf{a_3} = \frac{1}{(2k-6)!!}$. Then the row sum and the least eigenvalue of the matrix M are $(2k-1)(2k-3)-1$ and -1, respectively. Moreover, the only* *modules with eigenvalue equal to -1 are* $[2k-2,2]$, $[2k-4,4]$ *and* $[2k-4,2,2]$.

Proof. For $10 \leq k \leq 14$, similar to the proof of Theorem 3.3.6, by utilizing the complete character tables of the perfect matching association scheme [25, 31] we can find all the eigenvalues of the matrix M, and we see that the ratio bound holds with equality.

For the remainder of the proof assume that $k \geq 15$. Reviewing the linear system of equations, it follows that -1 is an eigenvalue of M (corresponding to the modules $[2k-2]$, $[2k-4,4]$, and $[2k-4,2,2]$). Denote the row sum by d_M; this is the linear combination of the degrees for matrices $A_{[2k]}$, $A_{[2k-2,2]}$, and $A_{[2k-4,4]}$; denote these by $d_{[2k]}$, $d_{[2k-2,2]}$, and $d_{[2k-4,4]}$, respectively. For the coefficients $\mathbf{a_1}$, $\mathbf{a_2}$, and $\mathbf{a_3}$, we calculate

$$\begin{aligned}
d_M &= \mathbf{a_1} d_{[2k]} + \mathbf{a_2} d_{[2k-2,2]} + \mathbf{a_3} d_{[2k-4,4]} \\
&= \frac{(2k-2)!!}{4(2k-6)!!} + \frac{k(2k-4)!!}{(2k-6)!!} + \frac{k(k-1)(2k-6)!!}{(2k-6)!!} \\
&= (2k-1)(2k-3) - 1.
\end{aligned}$$

Finally, we need to prove that all other eigenvalues of the matrix M are strictly greater than -1. Let $\{d_M^{(1)}, -1^{(m_1)}, -1^{(m_2)}, -1^{(m_3)}, \theta_4^{(m_4)}, \cdots, \theta_k^{(m_k)}\}$ be the spectrum of the matrix M, where the values m_i represent the multiplicity of the eigenvalues.

By Lemma 3.2.16, and the hook length formula we have,

$$m_1 = \frac{2k(2k-3)}{2},$$
$$m_2 = \frac{2k(2k-1)(2k-2)(2k-7)}{4!},$$
$$m_3 = \frac{2k(2k-1)(2k-4)(2k-5)}{12}.$$

As we defined, $M = \mathbf{a_1} A_{[2k]} + \mathbf{a_2} A_{[2k-2,2]} + \mathbf{a_3} A_{[2k-4,4]}$, the entry on the diagonal of the matrix M^2 is

$$\begin{aligned} M^2[i,i] &= \mathbf{a_1}^2 d_{[2k]} + \mathbf{a_2}^2 d_{[2k-2,2]} + \mathbf{a_3}^2 d_{[2k-4,4]} \\ &= \frac{(2k-2)!!}{(4(2k-6)!!)^2} + \frac{k(2k-4)!!}{((2k-6)!!)^2} + \frac{k(k-1)(2k-6)!!}{((2k-6)!!)^2} \\ &= \frac{13k^2-23k+2}{4(2k-6)!!}. \end{aligned}$$

It is well-known that the trace of any matrix is equal to the sum of its eigenvalues. Hence we have that the trace of M^2 is

$$\left(\frac{13k^2-23k+2}{4(2k-6)!!}\right)(2k-1)!! = d_M^2 + m_1 + m_2 + m_3 + \sum_{i=4}^{k} m_i \theta_i^2.$$

From the equation above, we obtain the following inequality for any eigenvalue θ_i of M,

$$|\theta_i| \leq \sqrt{\frac{(2k-1)!!}{(2k-6)!!}\left(\frac{13k^2-23k+2}{4}\right) - \left(18k^4 - 74k^3 + \frac{191}{2}k^2 - \frac{79}{2}k + 4\right)}\sqrt{\frac{1}{m_i}}. \tag{3.6}$$

To finish the proof, it is sufficient to prove that the term in the right-hand side of the above inequality is strictly less than 1; or equivalently

$$m_i > \frac{(2k-1)!!}{(2k-6)!!}\left(\frac{13k^2-23k+2}{4}\right) - \left(18k^4 - 74k^3 + \frac{191}{2}k^2 - \frac{79}{2}k + 4\right). \quad (3.7)$$

Let the partition $\lambda_i = [\lambda_{i_1}, \lambda_{i_2}, \cdots, \lambda_{i_\ell}]$ where $\lambda_{i_1} \geq \lambda_{i_2} \geq \cdots \geq \lambda_{i_\ell}$. Then there are 3 cases:

Case 1. Assume that $\lambda_{i_1} \geq k$.

First, consider the module $[2k-6, 6]$. Using the linear combination of the eigenvalues of the matrices $A_{[2k]}$, $A_{[2k-2,2]}$, and $A_{[2k-4,4]}$ corresponding to the module $[2k-6,6]$ it follows that the eigenvalue of M belonging to $[2k-6,6]$ (shown by $\theta_{[2k-6,6]}$) is

$$\begin{aligned}\theta_{[2k-6,6]} = &\frac{1}{4(2k-6)!!}\left(-3(2k-8)!!\right) + \\ &\frac{1}{(2k-6)!!}\left(-3(3k-10)(2k-10)!!\right) + \\ &\frac{1}{(2k-6)!!}\left(-3(9k^2-71k+140)(2k-12)!!\right) \\ = &\frac{-6(2k-12)!!}{(2k-6)!!}\left(8k^2-65k+130\right).\end{aligned}$$

This eigenvalue is greater than -1 for $k \geq 15$ (this can be checked with Maple software [22]).

Second, consider the modules $[2k-6,4,2]$ and $[2k-6,2,2,2]$. Using the hook length formula, we calculate the dimensions $m_{[2k-6,4,2]}$ and $m_{[2k-6,2,2,2]}$. If, in the right hand side of (3.7), we approximate the term $(2k-6)!!$ with $(2k-7)!!$, then the inequality holds for $m_{[2k-6,4,2]}$ with $k \geq 37$, and for $m_{[2k-6,2,2,2]}$ for $k \geq 63$. Using Maple [22] for the values of $m_{[2k-6,4,2]}$ for $15 \leq k \leq 36$, and for $m_{[2k-6,2,2,2]}$ with $15 \leq k \leq 62$; without the approximation; observe that (3.7) holds for these two modules.

Next consider the module $[2k-8, 8]$. By Lemma 3.2.16 the multiplicity of the corresponding eigenvalue is

$$\frac{(2k)(2k-1)(2k-2)(2k-3)(2k-4)(2k-5)(2k-6)(2k-15)}{8!}.$$

Thus, if in the right hand side of (3.7), we approximate the term $(2k-6)!!$ with $(2k-7)!!$, then the inequality holds for all $k \geq 20$. Using Maple for the values $15 \leq k \leq 19$, we have that (3.7) holds for this module.

Finally, using Theorem 3.2.15 and Lemma 3.2.16, for all modules with $\lambda_{i_1} \geq k$ and $\lambda_{i_1} \leq 2k-8$, the multiplicity m_i is greater than the multiplicity for the module $[2k-8, 8]$. Thus (3.7) holds for all remaining modules with $\lambda_{i_1} > k$.

Case 2. Assume that $\lambda_{i_1} < k$ and λ_i is primary.

Using Theorem 3.2.17 for $2k \geq 8$, we have

$$m(\lambda) \geq F(2k) = 2F(2k-1) \geq (2)\left(\frac{3}{2}\right) F(2k-2) \geq \cdots \geq 2(3^k). \qquad (3.8)$$

Approximating the term $(2k-6)!!$ with $(2k-7)!!$, in (3.6), for $k \geq 15$ we have

$$\theta_i^2 \leq \frac{104k^5 - 724k^4 + 1738k^3 + 595k - 46}{4m_i} \leq \frac{104k^5}{8(3^k)} \leq 1.$$

In fact, for $k \geq 1$, the term $(-724k^4 + 1738k^3 + 595k - 46)$ is always negative. Noting this along with (4.13) proves the second inequality above. The last inequality holds for all $k \geq 15$.

Case 3. Assume that $\lambda_{i_1} < k$ and λ_i is not primary.

We know that the degrees of a partition and its dual are the same, so $m(\lambda) = m(\lambda^*)$. Let $\lambda^* = (\lambda_1^*, \lambda_2^*, \cdots, \lambda_t^*)$. Since λ is an even partition, then $\lambda_1^* = \lambda_2^*$. Set f to be λ_1^*. Since λ_i is not primary, $\lambda^* \geq \lambda$. This means that λ^* is primary. If $f \geq k$, then $\lambda^* = [k, k]$ which is covered in Case 1. If $f < k$, this is covered by Case 2.

□

Recall that a canonical 2-intersecting set of perfect matchings is the set of all perfect matchings that contain the edges e_1 and e_2. The size of a canonical 2-intersecting set is $(2k-5)!!$. Theorems 3.3.6 and 3.3.7, along with the ratio bound (Theorem 3.1.1), show that the size of a 2-intersecting set of perfect matchings is no larger than $(2k-5)!!$. The ratio bound further implies that if S is a maximum 2-intersecting set and v_S is the characteristic vector of S, then $v_S - \frac{1}{(2k-1)(2k-3)}\mathbf{1}$ is a -1-eigenvector for $M_2(2k)$. Since the only irreducible representations of $\mathrm{Sym}(2k)$ that afford -1 as an eigenvalue are $[2k-2, 2]$, $[2k-4, 4]$, and $[2k-4, 2, 2]$, this implies that v_S is in $\mathrm{span}\{[2k], [2k-2, 2], [2k-4, 4], [2k-4, 2, 2]\}$. The next result gives a more convenient spanning set for this space. Define ν_{e_1,e_2} be the characteristic vector of all perfect matchings with the (disjoint) edges e_1 and e_2.

Theorem 3.3.8. *For $k \geq 4$, let*

$$V(2k) = \mathrm{span}\{\nu_{e_1,e_2} \,|\, e_1, e_2 \text{ edges in } K_{2k}\}$$

and

$$W(2k) = \mathrm{span}\{[2k], [2k-2,2], [2k-4,4], [2k-4,2,2]\}.$$

Then $W(2k) = V(2k)$.

Proof. Using the ratio bound for the weighted adjacency matrices given in Theorems 3.3.6 and 3.3.7, we see that the size of a maximum coclique in $M_2(2k)$ is $(2k-5)!!$, so all perfect matchings which have two fixed edges, say e_1 and e_2, form a maximum coclique. Thus, by Theorem 3.1.1, $\nu_{e_1,e_2} - \frac{(2k-5)!!}{(2k-1)!!}\mathbf{1}$ is an eigenvector for the least eigenvalue. The least eigenvalue is -1 and only the modules $[2k-2,2]$, $[2k-4,4]$, and $[2k-4,2,2]$ have -1 as their eigenvalues. So $\nu_{e_1,e_2} \in W(2k)$, and $V(2k) \subseteq W(2k)$. Denote the dimensions of the sets $W(2k)$ and $V(2k)$, by $D_W(2k)$ and $D_V(2k)$. Then, by Lemma 3.2.16, and more generally the hook length formula

$$D_W(2k) = 1 + \binom{2k}{2} - \binom{2k}{1} + \binom{2k}{4} - \binom{2k}{3} + \frac{(2k)(2k-1)(2k-4)(2k-5)}{12}.$$

For $4 \leq k \leq 11$, using GAP [12], we note $D_V(2k) = D_W(2k)$, thus $V(2k) = W(2k)$. For $k > 11$, we prove the same result by induction.

By summing all vectors in $V(2k+2)$, we obtain a multiple of the all ones vectors, so the module $[2k+2]$ is contained in $V(2k+2)$. Similarly, by summing all vectors with a fixed edge and subtracting an appropriate multiple of the all ones vector, we see the space $V(2k)$ contains the characteristic vector of the set of all perfect matchings that contain a fixed edge. Thus $V(2k+2)$ includes the module $[2k, 2]$ [16, Lemma 8.2].

For any set $S \subset \{1, 2, \ldots, 2k+2\}$ of size four the characteristic vector of all the perfect matchings in which the four elements of S appear as two independent edges is in $V(2k+2)$. By Theorem 3.2.14 these vectors are a spanning set for span$\{[2k+2], [2k, 2], [2k-2, 4]\}$. Thus each of these modules is contained in $V(2k)$.

Since the edges of K_{2k} are a subset of the edges form K_{2k+2}, the space $V(2k+2)$ contains a subspace isomorphic to $V(2k)$. By induction, the dimension of $V(2k)$ is

$$D(2k) = 1 + \binom{2k}{2} - \binom{2k}{1} + \binom{2k}{4} - \binom{2k}{3} + \frac{(2k)(2k-1)(2k-4)(2k-5)}{12}.$$

This is a lower bound on the dimension of $V(2k+2)$.

For $k \geq 11$,

$$D_V(2k) > D([2k+2]) + D([2k, 2]) + D([2k-2, 4])$$

We conclude that $V(2k+2)$ must include some of the space $[2k-2, 2, 2]$. But since these are irreducible G-modules and $V(2k+2)$ is invariant over G, this implies all of $[2k-4, 2, 2]$ is also contained in $V(2k+2)$. □

Putting these results together, we have our main result.

Theorem 3.3.9. *The size of the largest set of* 2*-intersecting perfect matchings in* K_{2k} *with* $k \geq 3$ *is* $(2k-5)!!$*. Further, if* S *is a set of* 2*-intersecting perfect matchings, the characteristic vector of* S *is a linear combination of the characteristic vectors of the canonically 2-intersecting sets of perfect matchings.*

Chapter 4

Set-Wise t-intersecting Perfect Matchings

In Chapter 3, we proved a version of the Erdős-Ko-Rado theorem for 2-intersecting perfect matchings. As we mentioned in Section 2.4, in [6], Ellis defined the concept of set-wise intersecting families for permutations. The motivation for this chapter is to generalize these two works to the family of set-wise t-intersecting perfect matchings of the complete graph K_{2k}. The approach we take here is similar to the second approach in Chapter 3. The results of this chapter are included in the paper [30].

Consider $\mathcal{S}_t(k, \ell)$, the set of canonically set-wise t-intersecting uniform set partitions. Recall that $\mathcal{U}_{\ell,k}$ is the set of all (ℓ, k)-partitions, and $u_{\ell,k}$ is the size of this set. We propose the following conjecture.

Conjecture 4.0.1. *The size of the largest set of set-wise t-intersecting uniform set partitions in $\mathcal{U}_{\ell,k}$ with $k \geq 2t$ is $|\mathcal{S}_t(k, \ell)| = u_{\ell,t}u_{\ell,k-t}$ and a set with maximum size is an orbit of the Young subgroup $Sym(\ell t) \times Sym(\ell k - \ell t)$.*

In particular, the above conjecture for perfect matchings in the complete graph K_{2k} indicates that if $k \geq 2t$, the size of a largest set of set-wise t-intersecting perfect matchings is $(2t-1)!!(2k-2t-1)!!$. In Section 4.2 we prove this conjecture for $t = 2$.

4.1 Derangement Graph for Set-Wise 2-Intersecting Perfect Matchings

In this section we define a graph $N_t(2k)$ for which finding the size of the largest cocliques is equivalent to finding the size of the largest set of set-wise t-intersecting perfect matchings. Then in Section 4.2 we construct a weighted adjacency matrix for the graph $N_2(2k)$ and we show that for this matrix, the ratio bound holds with equality. This is similar to the approach used in Section 3.3.2.

Definition 4.1.1. *For $t \leq \lfloor \frac{k}{2} \rfloor$, define the graph $N_t(2k)$ to be the graph with perfect matchings of the complete graph K_{2k} as its vertices. In this graph, two perfect matchings are adjacent if there is no partition of $2t$ as a sub-partition of their shape.*

For example, in $N_3(2k)$, two perfect matchings on $2k$ vertices are adjacent if there is no 6-cycle, no 4-cycle and 2-cycle pair, and no 2-cycle triple in their intersection. Denote the adjacency matrix of $N_t(2k)$ by $A_t(2k)$. Clearly, the graph $N_t(2k)$ is a graph in the perfect matching association scheme and we have that

$$A_t(2k) = \sum_{\lambda \vdash 2k} A_\lambda, \tag{4.9}$$

where the partitions λ contain no partition of $[2t]$ as a sub-partition. So the eigenvalues of $N_t(2k)$ are the sum of the eigenvalues of the matrices A_λ in (4.9). Similarly, a weighted adjacency matrix $B_t(2k)$ of the graph $N_t(2k)$ is in $\mathbb{C}[\mathcal{A}]$,

$$B_t(2k) = \sum_{\lambda \vdash 2k} a_\lambda A_\lambda, \tag{4.10}$$

where $A_\lambda \in \mathcal{A}$, partitions λ have no partition of $[2t]$ as a sub-partition and $a_\lambda \in \mathbb{C}$.

4.2 EKR Theorem for Set-wise 2-intersecting Perfect Matchings

We define the coefficients a_λ in (4.10), for which the ratio bound holds with equality for $B_t(2k)$, and hence for $N_t(2k)$, when $t = 2$. Here for the simplicity, let $\mathcal{S}_t(2k) = \mathcal{S}_t(k, 2)$. Therefore, our approach is to show that in the ratio bound we have that

$$1 - \frac{d}{\tau} = \frac{\mid V(N_t(2k)) \mid}{\mid \mathcal{S}_t(2k) \mid}.$$

Thus for $t = 2$, the set of all perfect matchings with a fixed set of $2t$ elements in exactly t edges is a maximum family of set-wise t-intersecting perfect matchings. If we can find a set of coefficients for the graph $B_2(2k)$ in (4.10), such that the row sum and the least eigenvalue are $\frac{(2k-1)(2k-3)}{3} - 1$ and -1 respectively, then the ratio bound will hold with equality,

$$| \mathcal{S}_2(2k) |= \frac{|V(X)|}{1-\frac{d}{\tau}} = \frac{(2k-1)!!}{1-\frac{\frac{(2k-1)(2k-3)-1}{3}}{-1}} = 3(2k-5)!!.$$

For $k \geq 3$, set all coefficients a_λ in $B_2(2k)$ to be 0, except $a_{[2k]}$ and $a_{[2k-2,2]}$. Then we have

$$\widehat{B}_2(2k) = a_{[2k]}A_{[2k]} + a_{[2k-2,2]}A_{[2k-2,2]}.$$

Using the general formulas for the eigenvalues of $A_{[2k]}$ and $A_{[2k-2,2]}$ in the character table 3.2.2, we construct a system of linear equations in which equations correspond to the irreducible modules $[2k]$, $[2k-2,2]$, and $[2k-4,4]$,

$$\begin{aligned}
(2k-2)!!\mathbf{a}_{[\mathbf{2k}]} + \quad & (k)(2k-4)!!\mathbf{a}_{[\mathbf{2k-2,2}]} = \tfrac{(2k-1)(2k-3)}{3} - 1,\\
-(2k-4)!!\mathbf{a}_{[\mathbf{2k}]} + \quad & \tfrac{1}{2}(2k-4)!!\mathbf{a}_{[\mathbf{2k-2,2}]} = -1,\\
-(2k-6)!!\mathbf{a}_{[\mathbf{2k}]} - (5k-12) & (2k-8)!!\mathbf{a}_{[\mathbf{2k-2,2}]} = -1.
\end{aligned}$$

By considering the second and the third equations above, we obtain the unique solutions $a_{[2k]} = \frac{k}{3(2k-4)!!}$ and $a_{[2k-2,2]} = \frac{(2k-6)}{3(2k-4)!!}$, for all $k \geq 4$; which satisfies the first equation above. Thus, the row sum of $\widehat{B}_2(2k)$ would be $d_{\hat{B}} = \frac{(2k-1)(2k-3)}{3} - 1$. To finish this argument, we prove that every other eigenvalue of $\widehat{B}_2(2k)$ is between -1 and $d_{\hat{B}}$.

Theorem 4.2.1. *For $k \geq 7$, let*

$$\widehat{B}_2(2k) = \frac{k}{3(2k-4)!!} A_{[2k]} + \frac{(2k-6)}{3(2k-4)!!} A_{[2k-2,2]}.$$

Then the row sum and the least eigenvalue of the matrix $\widehat{B}_2(2k)$ are $\frac{(2k-1)(2k-3)}{3} - 1$ and -1, respectively. Furthermore, the only modules with eigenvalue equal to -1 are $[2k-2,2]$ and $[2k-4,4]$, and all other eigenvalues are in $(-1, \frac{(2k-1)(2k-3)}{3} - 1)$.

Proof. In [31], Srinivasan implemented a Maple program to compute the complete character table of the perfect matching association scheme for all $k \leq 40$. So by utilizing the complete character table for $3 \leq k \leq 12$, we find the eigenvalues of $\widehat{B}_2(2k)$, which will verify that equality holds in the ratio bound. For the remainder of the proof, let $k \geq 12$. Let $\{d_{\hat{B}}^{(1)}, -1^{(m_1)}, -1^{(m_2)}, \theta_3^{(m_3)}, \ldots, \theta_\ell^{(m_\ell)}\}$ be the spectrum of the matrix $\widehat{B}_2(2k)$, where the values m_i are the multiplicities of the eigenvalues.
By Lemma 3.2.16, the multiplicity of any module $[2k-\ell, \ell]$; denoted by $m([2k-\ell, \ell])$; can be calculated by $m([2k-\ell, \ell]) = \binom{2k}{\ell} - \binom{2k}{\ell-1}$. Hence,

$$m_1 = \frac{2k(2k-3)}{2},$$
$$m_2 = \frac{2k(2k-1)(2k-2)(2k-7)}{4!}.$$

Now consider the row sum of the matrix $(\widehat{B})^2$. The main diagonal entries of $(\widehat{B})^2$ are given by

$$\begin{aligned}\widehat{B}^2(i,i) = d_{\widehat{B}^2} =& a_{[2k]}^2 d_{[2k]} + a_{[2k-2,2]}^2 d_{[2k-2,2]}\\ =& \frac{k^2}{9(2k-4)!!^2}(2k-2)!! + \frac{(2k-6)^2}{9(2k-4)!!^2}k(2k-4)!!\\ =& \frac{k(6k^2-26k+36)}{9(2k-4)!!},\end{aligned}$$

where $d_{[2k]}$ and $d_{[2k-2,2]}$ are the degrees of the matrices $A_{[2k]}$ and $A_{[2k-2,2]}$, respectively. It is also well-known that the trace of a matrix is the sum of its eigenvalues. So,

$$\left(\frac{k(6k^2-26k+36)}{9(2k-4)!!}\right)(2k-1)!! = d_{\widehat{B}}^2 + m_1 + m_2 + \sum_{i=3}^{k} m_i\theta_i^2.$$

Hence,

$$\begin{aligned}& k\left(6k^2-26k+36\right)\frac{(2k-1)!!}{9(2k-4)!!} - \left(\frac{(2k-1)(2k-3)-3}{3}\right)^2\\ &- \frac{2k(2k-3)}{2} - \frac{2k(2k-1)(2k-2)(2k-7)}{4!} = \sum_{i=3}^{k} m_i\theta_i^2.\end{aligned}$$

This means that for every θ_i, $3 \leq i \leq \ell$, we have that,

$$k\left(6k^2-26k+36\right)\frac{(2k-1)!!}{9(2k-4)!!} - \frac{k(11k-25)(2k-1)(2k-3)}{18} \geq m_i\theta_i^2. \qquad (4.11)$$

Next we show that any eigenvalue θ_i is greater than -1, for $3 \leq i \leq \ell$. Let θ_3 be the eigenvalue of $\widehat{B}_2(2k)$ corresponding to the module $[2k-4,2,2]$. Then, θ_3 is the linear combination of the eigenvalues of the matrices $A_{[2k]}$ and $A_{[2k-2,2]}$ in the character table 3.2.2, for the module $[2k-4,2,2]$.

$$\theta_3 = \frac{k}{3(2k-4)!!}(2(2k-6)!!) + \frac{(2k-6)}{3(2k-4)!!}(-(2k-6)!!) = \frac{1}{(k-2)}.$$

For the other eigenvalues, it is sufficient to show that the following inequality holds.

$$k\left(6k^2-26k+36\right)\frac{(2k-1)!!}{9(2k-4)!!}-\frac{k(11k-25)(2k-1)(2k-3)}{18}<m_i, \tag{4.12}$$

since this, along with the equation (4.11) shows that $\theta_i^2<1$. Let m_i be the multiplicity of the λ_i-module, where $\lambda_i=[\lambda_{i_1},\lambda_{i_2},\cdots,\lambda_{i_\ell}]$, and $\lambda_{i_1}\geq\lambda_{i_2}\geq\cdots\geq\lambda_{i_\ell}$. There are 2 cases:

Case 1. Suppose that $\lambda_{i_1}\geq k$.

Consider the module $[2k-6,6]$. Then,

$$m_i=\binom{2k}{6}-\binom{2k}{5}=\frac{(2k)(2k-1)(2k-2)(2k-3)(2k-4)(2k-11)}{6!}.$$

By substituting the formula for m_i in inequality (4.12), and approximating the term $(2k-4)!!$ by $(2k-5)!!$, the inequality holds for all $k\geq 34$ (this can be confirmed with Maple software). Also, using Maple for the values of $12\leq k\leq 33$, we can see that inequality (4.12) holds. By Theorem 3.2.15, and the fact that $m([2k-\ell,\ell])=\binom{2k}{\ell}-\binom{2k}{\ell-1}$, the multiplicity m_i is greater than or equal to the multiplicity of the module $[2k-6,6]$, for all modules with $k\leq\lambda_{i_1}\leq 2k-6$. Hence, inequality (4.12) holds for all modules with $\lambda_{i_1}>k$.

Case 2. Suppose that $\lambda_{i_1}<k$.

If λ_i is primary, by Theorem 3.2.17, For $2k\geq 8$, we have that,

$$m(\lambda)\geq F(2k)=2F(2k-1)\geq(2)\left(\frac{3}{2}\right)F(2k-2)\geq\cdots\geq 3^{k-4}F(8)\geq 403200(3^{k-4}). \tag{4.13}$$

Using (4.13), and approximating the term $(2k-4)!!$ by $(2k-5!!)$ in (4.12), it is sufficient to show that,

$$\frac{48k^5 - 348k^4 + 928k^3 - 965k^2 + 921k}{18} < 403200(3^{k-4}).$$

The expression $-348k^4 + 928k^3 - 965k^2 + 921k$ in the numerator of the fraction on the left side of the above inequality is negative for all $k \geq 2$, so it is sufficient to check when $\frac{48k^5}{18} < 403200(3^{k-4})$. In fact, this inequality is true for all k; therefore, for all primary partitions λ_i with $\lambda_{i_1} < k$, the inequality (4.12) holds.

Now assume that $\lambda_{i_1} < k$ and λ_i is not primary. The proof of this part follows from the proof of Case 3 in Theorem 4.11 in [10]. The dual of λ_i; $\lambda^* = (\lambda_1^*, \lambda_2^*, \cdots, \lambda_t^*)$; is primary and $\lambda_1^* = \lambda_2^*$. Note that $m(\lambda_i) = m_{\lambda^*}$. If $\lambda_1^* \geq k$, then λ^* is $[k, k]$, which is covered by Case 1. For $\lambda_1^* < k$, we just proved the result in Case 2.

□

4.3 Conjecture on Set-Wise 3-intersecting Perfect Matchings

In this section, we construct a weighted adjacency matrix for the graph $N_3(2k)$, and conjecture that the ratio bound holds with equality for this matrix. For $t = 3$, if we can find a set of coefficients in (4.10), so that the weighted adjacency matrix has degree $d = \frac{(2k-1)(2k-3)(2k-5)}{15} - 1$ and least eigenvalue $\tau = -1$, then $1 - \frac{d}{\tau} = \frac{|V(N_3(2k))|}{|\mathcal{S}_3(2k)|}$; thus, the ratio bound would hold with equality. Let

$$\widehat{B}_3(2k) = a_{[2k]}A_{[2k]} + a_{[2k-2,2]}A_{[2k-2,2]} + a_{[2k-4,2,2]}A_{[2k-4,2,2]}.$$

We consider a system of linear equations for $\hat{B}_3(2k)$; similar to the one for $t = 2$. In this system the equations correspond to the irreducible modules $[2k-2,2]$, $[2k-4,4]$, and $[2k-6,6]$; we want weights so the corresponding eigenvalues equal to -1.

$$\begin{aligned}
\tfrac{(2k)!!}{2k}\mathbf{a}_{[\mathbf{2k}]} + {} & \tfrac{(2k)!!}{2(2k-2)}\mathbf{a}_{[\mathbf{2k-2,2}]} + {} & \tfrac{(2k)!!}{8(2k-4)}\mathbf{a}_{[\mathbf{2k-4,2,2}]} &= d,\\
-(2k-4)!!\mathbf{a}_{[\mathbf{2k}]} + {} & \tfrac{(2k-4)!!}{2}\mathbf{a}_{[\mathbf{2k-2,2}]} + {} & \tfrac{(3k-2)(2k-6)!!}{4}\mathbf{a}_{[\mathbf{2k-4,2,2}]} &= -1,\\
-(2k-6)!!\mathbf{a}_{[\mathbf{2k}]} - {} & (5k-12)(2k-8)!!\mathbf{a}_{[\mathbf{2k-2,2}]} - {} & \tfrac{(k+3)(2k-8)!!}{4}\mathbf{a}_{[\mathbf{2k-4,2,2}]} &= -1,\\
-3(2k-8)!!\mathbf{a}_{[\mathbf{2k}]} - {} & (3k-10)(2k-10)!!\mathbf{a}_{[\mathbf{2k-2,2}]} - {} & \tfrac{-3(13k^2-101k+190)(2k-12)!!}{2}\mathbf{a}_{[\mathbf{2k-4,2,2}]} &= -1,
\end{aligned}$$

For all $k \geq 6$, the unique solutions of this system are $\mathbf{a}_{[\mathbf{2k}]} = \frac{(k-3)(7k-10)}{30(2k-4)!!}$, $\mathbf{a}_{[\mathbf{2k-2,2}]} = \frac{-2(k^2-10k+15)}{15(2k-4)!!}$, and $\mathbf{a}_{[\mathbf{2k-4,2,2}]} = \frac{2(k-5)}{5(2k-6)!!}$.

Conjecture 4.3.1. *For $k \geq 11$, let*

$$\widehat{B}_3(2k) = A_{[2k]}\frac{(k-3)(7k-10)}{30(2k-4)!!} + A_{[2k-2,2]}\frac{-2(k^2-10k+15)}{15(2k-4)!!} + A_{[2k-4,2,2]}\frac{2(k-5)}{5(2k-6)!!}.$$

Then the row sum and the least eigenvalue of the matrix $\widehat{B}_3(2k)$ are $\frac{(2k-1)(2k-3)(2k-5)}{15} - 1$ and -1, respectively. Furthermore, the only modules with eigenvalue equal to -1 are $[2k-2,2]$, $[2k-4,4]$, and $[2k-6,6]$.

Similar to the proof of Theorem 4.2.1, for all modules except $[2k-6,4,2]$ and $[2k-6,2,2,2]$, it can be shown that the associated eigenvalues are between $\frac{(2k-1)(2k-3)(2k-5)}{15}-1$ and -1. We also conjecture that this is true for modules $[2k-6,4,2]$ and $[2k-6,2,2,2]$, as we confirmed this with Maple [31] for some values of k; but finding the exact formulas of the corresponding eigenvalues of this modules using the quotient graphs is quite complicated. Using symmetric functions and character theory might be an approach for results along these lines in the future [25].

Chapter 5

Erdős-Ko-Rado Theorem For Partially 2-intersecting Uniform Set Partitions

As stated at the end of Section 2.4, Meagher and Moura [23] conjectured that for $t \leq \ell$, if $\mathcal{P} \subset \mathcal{U}_{\ell,k}$ is a set of partially t-intersecting uniform partitions, then $|\mathcal{P}| \leq \binom{k\ell-t}{\ell-t} u_{\ell,k-1}$. In this chapter, we show for k sufficiently large, a set of partially 2-intersecting (ℓ, k)-partitions is no larger than the size of the set of all (ℓ, k)-partitions in which a block contains a fixed pair. Consequently we prove the Meagher-Moura conjecture for $t = 2$, and for all $\ell \geq 4$ provided that k is sufficiently large. Our approach is to define a graph in which the cocliques are equivalent to partially 2-intersecting (ℓ, k)-partitions from $\mathcal{U}_{\ell,k}$. Then we apply algebraic methods to find the size of a maximum coclique in the graph. This approach is similar to the technique applied in Chapters 3 and 4. The content of this chapter is from the paper [24], which was a joint work with Meagher and Stevens.

Definition 5.0.1. *Define $X_{\ell,k}$ to be the graph with $\mathcal{U}_{\ell,k}$ as its vertex set, in which two partitions P and Q are adjacent if every pair of blocks, one from P and one from Q, have at most 1 element in common.*

The group $\mathrm{Sym}(k\ell)$ acts transitively on the vertices of $X_{\ell,k}$ and preserves the edges. This means that $X_{\ell,k}$ is vertex transitive and regular. We will denote the degree by $d_{\ell,k}$, or simply d when the context is clear.

For any distinct $i, j \in \{1, \dots, \ell k\}$, let $S_{i,j}$ be the subset of partitions in $\mathcal{U}_{\ell,k}$ for which the elements i and j are in the same block. Then $S_{i,j}$ is a coclique in the graph $X_{\ell,k}$ and the size of $S_{i,j}$ is

$$\frac{1}{(k-1)!}\binom{k\ell-2}{\ell-2}\binom{k\ell-\ell}{\ell}\cdots\binom{\ell}{\ell}.$$

The main goal in this paper is to prove, using the ratio bound, that $S_{i,j}$ is a maximum coclique in $X_{\ell,k}$. For the ratio bound to hold with equality, we need to prove if τ is the least eigenvalue of $X_{\ell,k}$, then

$$1 - \frac{d_{\ell,k}}{\tau} = \frac{u_{\ell,k}}{|S_{i,j}|} = \frac{k\ell-1}{\ell-1}.$$

Thus we need to prove two facts: first that $\tau = -\frac{d_{\ell,k}(\ell-1)}{\ell(k-1)}$ is an eigenvalue of $X_{\ell,k}$; and second that τ is the least eigenvalue of $X_{\ell,k}$.

In the next section, we calculate three of the eigenvalues of $X_{\ell,k}$; one of these eigenvalues is τ as presented above. To prove that τ is the least eigenvalue, in Section 5.3, we show if there is another eigenvalue, strictly smaller than τ, then its multiplicity must be bounded by a function that includes the ratio $u_{\ell,k}/d_{\ell,k}$. In Section 5.2, we show that the limit of ratio $u_{\ell,k}/d_{\ell,k}$ is finite as $k \to \infty$. This gives a simple upper bound on $u_{\ell,k}/d_{\ell,k}$ for all sufficiently large k. In Section 5.3 we show that no such eigenvalues exist. This proves the Meagher-Moura Conjecture with $t = 2$, for all values of ℓ, provided that k is sufficiently large.

5.1 Eigenvalues of $X_{\ell,k}$ with $\ell \geq 3$

In this section we will find three of the eigenvalues of $X_{\ell,k}$. For ease of notation, we will denote the irreducible representation of χ_λ by the λ-module. Also, the number of vertices in $X_{\ell,k}$, which is equal to $u_{\ell,k}$, will be denoted simply by v and the degree of the graph $X_{\ell,k}$ will be simply written as d, rather than $d_{\ell,k}$.

As we saw in Chapter 4, any subgroup $H \leq \mathrm{Sym}(k\ell)$ acts on the vertices of $X_{\ell,k}$ and the orbits of this action form an equitable partition. From any equitable partition, we can form a quotient graph and the eigenvalues of this quotient graph will be eigenvalues of the $X_{\ell,k}$. The trivial case is $H = \mathrm{Sym}(k\ell)$, since this group is transitive, the equitable partition has all the vertices of $X_{\ell,k}$ in a single part. The quotient graph for this is the 1×1 matrix with the single entry d. The eigenvalue of this matrix is simply d, and the eigenvector is the all ones vector and the eigenspace is isomorphic to the trivial representation of $\mathrm{Sym}(k\ell)$. So d belongs to the $[k\ell]$-module.

Since the subgroup $\mathrm{Sym}([k\ell - 1, 1])$ has only one orbit on the vertices of $X_{\ell,k}$, the next subgroup we consider is the Young subgroup $\mathrm{Sym}([k\ell - 2, 2])$, considered as the stabilizer of the set $\{1, 2\}$. This subgroup is not transitive on the partitions, it has exactly 2 orbits: S_1 the set of all partitions that have 1 and 2 in the same block, and S_2 the set of all partitions in which 1 and 2 are in different blocks. The first orbit, S_1 is a coclique in $X_{\ell,k}$ so the quotient matrix for this partition has the form

$$\begin{pmatrix} 0 & d \\ -\tau & d+\tau \end{pmatrix}.$$

The eigenvalues of this quotient graph are d and τ. We can calculate the value of τ by counting edges between S_1 and S_2. Since S_1 is a coclique, each vertex in S_1 is adjacent to d vertices in S_2, and each vertex in S_2 is adjacent to $-\tau$ vertices in S_1. Using the sizes of S_1 and S_2, we have that the number of edges between S_1 and S_2 is equal to

$$|S_1|d = \binom{k\ell - 2}{\ell - 2} u_{\ell,k-1} d$$

and also is equal to

$$|S_2|(-\tau) = \binom{k\ell - 2}{\ell - 1}\binom{k\ell - \ell - 1}{\ell - 1} u_{\ell,k-2}(-\tau).$$

Thus

$$\tau = -\frac{(\ell - 1)d}{\ell(k - 1)} \tag{5.14}$$

is a second eigenvalue for $X_{\ell,k}$. By [24], since this eigenvalue arises from the action of

Sym$[k\ell-2,2]$, it belongs to a module that is common between the two representations

$$ind_{\mathrm{Sym}(k\ell)}(1_{\mathrm{Sym}(\ell)\wr\mathrm{Sym}(k)}) \text{ and } ind_{\mathrm{Sym}(k\ell)}(1_{\mathrm{Sym}([k\ell-2,2])}).$$

Thus it belongs to the module $[k\ell-2,2]$, as this is the only common module, other than the trivial module, and must have multiplicity at least $\binom{k\ell}{2}-\binom{k\ell}{1}$. (A second irreducible module could also have τ as the eigenvalue belonging to it, so the multiplicity could be higher.)

Lemma 5.1.1. *For integers k and ℓ, with $k,\ell \geq 2$, $\tau = -\frac{(\ell-1)d}{\ell(k-1)}$ is an eigenvalue of $X_{\ell,k}$ with multiplicity at least $\binom{k\ell}{2}-\binom{k\ell}{1}$.*

Next we will consider the Young subgroup Sym$([k\ell-3,3])$, thought of as the group that stabilizes the set $\{1,2,3\}$. The action of this subgroup on $\mathcal{U}_{\ell,k}$ has 3 orbits: T_1, the set of all partitions with $1,2,3$ in the same block; T_2, the set of all partitions in which $1,2,3$ are in exactly two different blocks; and T_3, the set of all partitions in which $1,2,3$ are in three different blocks. Any vertex in T_1 is adjacent only to vertices in T_3. Similarly, a vertex in T_2 can be adjacent to vertices in T_2 and T_3. The quotient graph for this equitable partition is

$$M = \begin{pmatrix} 0 & 0 & d \\ 0 & a & d-a \\ b & c & d-b-c \end{pmatrix}.$$

where a,b,c are all non-negative. The eigenvalues for this quotient graph will be the eigenvalues that belong to modules that are both the decomposition of

$ind_{\mathrm{Sym}(k\ell)}(1_{\mathrm{Sym}([k\ell-3,3])})$ and the decomposition of $ind_{\mathrm{Sym}(k\ell)}(1_{\mathrm{Sym}(\ell)\wr\mathrm{Sym}(k)})$. Thus the eigenvalues will belong to the $[k\ell]$, $[k\ell-2,2]$ and $[k\ell-3,3]$ modules. We have already seen that the eigenvalue for $[k\ell]$ is d, and the eigenvalue for $[k\ell-2,2]$ is τ. We will denote the eigenvalue belonging to $[k\ell-3,3]$ by θ.

Since the trace of the matrix is the sum of the eigenvalues we have that

$$d + a - b - c = d + \tau + \theta. \tag{5.15}$$

The number of edges between T_1 and T_3 is equal to

$$d|T_1| = d\binom{k\ell-3}{\ell-3}u_{\ell,k-1},$$

and is also equal to

$$b|T_3| = b\binom{k\ell-3}{\ell-1}\binom{k\ell-\ell-2}{\ell-1}\binom{k\ell-2\ell-1}{\ell-1}u_{\ell,k-3}.$$

Setting these equations equal to each other, then expanding the binomial coefficients and rearranging the terms yields

$$\frac{(\ell-1)(\ell-2)}{\ell^2(k-1)(k-2)}d = b.$$

Replacing $d = -\frac{\ell(k-1)}{\ell-1}\tau$ shows that

$$b = -\frac{(\ell-1)(\ell-2)}{\ell^2(k-1)(k-2)}\frac{\ell(k-1)}{(\ell-1)}\tau = -\frac{\ell-2}{\ell(k-2)}\tau. \tag{5.16}$$

Substituting the expression from (5.16) into Equation 5.15 produces the following formula for θ

$$\theta = a + \frac{\ell - 2}{\ell(k-2)}\tau - c - \tau = a - c + \frac{(\ell-2) - \ell(k-2)}{\ell(k-2)}\tau. \tag{5.17}$$

Similarly, counting the number of edges between T_2 and T_3 yields

$$3\binom{k\ell-3}{\ell-2}\binom{k\ell-\ell-1}{\ell-1}u_{\ell,k-2}(d-a) = \binom{k\ell-3}{\ell-1}\binom{k\ell-\ell-2}{k-1}\binom{k\ell-2\ell-1}{\ell-1}u_{\ell,k-3}(c)$$

Again, expanding the binomial coefficients and rearranging shows that

$$a = d - \frac{(k-2)\ell}{3(\ell-1)}c.$$

The characteristic polynomial of M is

$$x^3 + (-a+b+c-d)x^2 + (-ab+ad-bd-cd)x + abd.$$

Substituting in the values we have computed for b and c, and using the fact that τ is a root of the characteristic polynomial we get

$$a = \frac{2(\ell-1)}{\ell(k-1)}d. \tag{5.18}$$

From this we can compute that

$$c = \frac{3(k\ell-3\ell+2)(\ell-1)}{k^2(k-1)(k-2)}d. \tag{5.19}$$

Lemma 5.1.2. *For integers k and ℓ, with $k, \ell \geq 3$,*

$$\theta = \frac{2(\ell-1)(\ell-2)d}{\ell^2(k-1)(k-2)}$$

is an eigenvalue of $X_{\ell,k}$ with multiplicity at least $\binom{k\ell}{3} - \binom{k\ell}{2}$.

Proof. From Equations (5.17), (5.18) and (5.19), we can calculate that

$$\theta = \frac{2(\ell-1)(\ell-2)d}{\ell^2(k-1)(k-2)}. \tag{5.20}$$

From the comments above, $\theta = \frac{2(\ell-1)(\ell-2)d}{\ell^2(k-1)(k-2)}$ is the eigenvalue belonging to the unique $[k\ell-3,3]$-module in $ind_{\mathrm{Sym}(k\ell)}(1_{\mathrm{Sym}(\ell)\wr\mathrm{Sym}(k)})$. Since the dimension of the irreducible representation of $[k\ell-3,3]$ is $\binom{k\ell}{3}-\binom{k\ell}{2}$, the multiplicity of θ is at least $\binom{k\ell}{3}-\binom{k\ell}{2}$. $\square$

5.2 Bounds on the Degree of $X_{\ell,k}$

In this section we determine a lower bound on the degree of $X_{\ell,k}$ for all sufficiently large k. If P and Q are two partitions that are adjacent in $X_{\ell,k}$, then the meet table of P and Q is a $k\times k$ matrix with entries either 0 or 1, and further, the entries in each row and column in the meet table sum to ℓ. We define $\mathcal{M}_{\ell,k}$ to be the set of all such meet tables, so all $k\times k$ matrices with entries either 0 or 1, and row and column sums equal to ℓ. To find the degree of $X_{\ell,k}$, we first state a result on the number of such meet tables. Next, for a fixed partition P and a meet table $M\in\mathcal{M}_{\ell,k}$, we count the number of partitions Q for which the meet table of P and Q is M.

Bender [2] determined the asymptotic cardinality of $\mathcal{M}_{\ell,k}$.

Theorem 5.2.1 ([2]). *For positive integers k,ℓ*

$$\lim_{k\to\infty}|\mathcal{M}_{\ell,k}| = \frac{(k\ell)!}{(\ell!)^{2k}}e^{-\frac{(\ell-1)^2}{2}}.$$

To get a lower bound on $d_{\ell,k}$, we fix a partition P in $\mathcal{U}_{\ell,k}$, then for each $M \in \mathcal{M}_{\ell,k}$, we will count the number of partitions Q so that the meet table of P and Q is M, then we use Theorem 5.2.1 to bound the size of $\mathcal{M}_{\ell,k}$.

Lemma 5.2.2. *For positive integers k, ℓ with $\ell \leq k$,*

$$d_{\ell,k} = \frac{\ell!^k}{k!} |\mathcal{M}_{\ell,k}|.$$

Proof. Fix a partition $P \in \mathcal{U}_{\ell,k}$. Define a bipartite multigraph with the vertices in one part the set $\mathcal{M}_{\ell,k}$, and the vertices in the other part the neighbourhood of P in $X_{\ell,k}$. Two vertices M and Q are adjacent if the meet table of P and Q is M. By counting the number of edges in this graph in two ways, we will determine the size of the neighbourhood of P in terms of $|\mathcal{M}_{\ell,k}|$.

For any $M \in \mathcal{M}_{\ell,k}$, with $M = [m_{i,j}]$ assume that row i corresponds to the block $P_i \in P$. Construct a partition $Q = \{Q_1, Q_2, \ldots, Q_k\}$ so that the block Q_j corresponds to column j of M and $|P_i \cap Q_j| = m_{i,j}$. Since the entries of a row in M are either 0 or 1, and sum to ℓ, there are $\ell!$ ways to select how the elements from P_i will be distributed among the blocks of Q. So for each meet table M, there are $\ell!^k$ partitions Q that can be constructed this way. It is possible that some of these partitions are equal, once the blocks are reordered, so this is a multigraph.

For every partition Q in the neighbourhood of P, there are $k!$ ways to order the blocks of Q; once the blocks are ordered; the meet table for P and Q is uniquely defined. In the bipartite graph, Q is adjacent to each of these tables in the graph (again, these tables may not be distinct, so the graph might be a multigraph). The degree of every vertex Q is $k!$. Thus we have that the number of edges in the multigraph is

$$k!d_{\ell,k} = \sum_{M \in \mathcal{M}_{\ell,k}} \ell!^k,$$

and the result follows. □

Using Theorem 5.2.1 we have the asymptotic size of $d_{\ell,k}$.

Corollary 5.2.3. *For a fixed integer ℓ with $\ell \geq 2$,*

$$\lim_{k \to \infty} \frac{u_{\ell,k}}{d_{\ell,k}} = e^{\frac{(\ell-1)^2}{2}}.$$

Proof. This follows from the value of $u_{\ell,k}$ given in Equation (2.3) and from the fact that

$$\lim_{k \to \infty} d_{\ell,k} = \frac{(k\ell)!}{(\ell!)^k k!} e^{\frac{-(\ell-1)^2}{2}}.$$

□

Thus for every $\epsilon > 0$, there exists a k' such that for all $k \geq k'$,

$$\frac{u}{d} \leq e^{\frac{(\ell-1)^2}{2}} + \epsilon.$$

5.3 A Bound on the Multiplicities of Eigenvalues with Large Absolute Value

In Section 5.1 we found three eigenvalues, d, τ, and θ of $X_{\ell,k}$. The ratio $\frac{d}{\tau} = \frac{\ell(1-k)}{\ell-1}$, so

$$\frac{|V(X_{\ell,k})|}{1-\frac{d}{\tau}} = \frac{|V(X_{\ell,k})|}{1-\frac{\ell(1-k)}{\ell-1}} = u_{\ell,k-1}.$$

This is exactly the size of a set of canonically 2-intersecting (ℓ,k)-partitions. If we can show that τ is the least eigenvalue of $X_{\ell,k}$, then the ratio bound implies that these are cocliques of maximum size. In this section we prove that if $X_{\ell,k}$ has an eigenvalue λ with $\lambda^2 > \tau^2$, then there is a bound on the multiplicity of λ. Let

$$\{d^{(1)}, \tau^{(m_\tau)}, \theta^{(m_\theta)}, \lambda_2^{(m_2)}, \ldots, \lambda_j^{(m_j)}\}$$

be the spectrum of the matrix $X_{\ell,k}$, where the values m_i represent the multiplicities of the eigenvalues. By squaring A and taking the trace, we have

$$vd = d^2 + m_\tau \tau^2 + m_\theta \theta^2 + \sum_{i=2}^{j} m_i \lambda_i^2.$$

Hence for every $2 \leq i \leq j$ we have

$$vd - d^2 - m_\tau \tau^2 - m_\theta \theta^2 \geq m_i \lambda_i^2.$$

Assume λ_i is an eigenvalue of $X_{\ell,k}$ with $\lambda_i^2 > \tau^2$, and also that λ_i is not the eigenvalue belonging to the $[k\ell]$, $[k\ell-2,2]$ or $[k\ell-3,3]$ modules, then

$$\frac{vd - d^2 - m_\tau \tau^2 - m_\theta \theta^2}{\tau^2} \geq m_i.$$

Expanding θ using Equation (5.20) in the above equation produces the following equation

$$\left(\frac{v}{d}-1\right)\frac{\ell^2(k-1)^2}{(\ell-1)^2}-m_\theta\frac{4(\ell-2)^2}{\ell^2(k-2)^2}-m_\tau \geq m_i.$$

Further, by Lemmas 5.1.1 and 5.1.2, it is known that $m_\tau \geq \binom{k\ell}{2}-\binom{k\ell}{1}$ and $m_\theta \geq \binom{k\ell}{3}-\binom{k\ell}{2}$, so this bound becomes

$$\left(\frac{v}{d}-1\right)\frac{\ell^2(k-1)^2}{(\ell-1)^2}-\frac{(k\ell)(k\ell-1)(k\ell-5)}{6}\frac{4(\ell-2)^2}{\ell^2(k-2)^2}-\frac{(k\ell)(k\ell-3)}{2} \geq m_i.$$

Our next step is to show that this upper bound on m_i is smaller than $\binom{k\ell}{3}-\binom{k\ell}{2}$. This will lead to a contradiction since we have assumed that λ does not belong to any of the $[k\ell]$, $[k\ell-2,2]$, and $[k\ell-3,3]$ modules. In other words, we need to prove that

$$\frac{v}{d}-1<\frac{k(\ell-1)^2}{6\ell^3(k-1)^2(k-2)^2}\left(\ell^2(k-2)^2(k\ell-4)(k\ell+1)+4(\ell-2)^2(k\ell-1)(k\ell-5)\right). \tag{5.21}$$

This follows from Corollary 5.2.3.

In [24], the following theorem has been proved.

Theorem 5.3.1. *[24] Assume $k\ell \geq 13$. Then the only partitions in the decomposition of $ind_{\mathrm{Sym}(k\ell)}(1_{\mathrm{Sym}(\ell)\wr\mathrm{Sym}(k)})$ with dimension less than or equal to $\binom{k\ell}{3}-\binom{k\ell}{2}$ are*

$$\chi_{[k\ell]},\quad \chi_{[k\ell-2,2]},\quad \chi_{[k\ell-3,3]}.$$

Theorem 5.3.1 will be used in the proof of the following theorem.

Theorem 5.3.2. *Fix an integer $\ell \geq 3$. For k sufficiently large, the largest set of partially 2-intersecting uniform (k, ℓ)-partitions has size*

$$\binom{k\ell - 2}{\ell - 2} u_{\ell,k-1}.$$

Proof. For any distinct $i, j \in \{1, \dots, k\ell\}$, the set $S_{i,j}$ of all (ℓ, k)-partitions with i and j in the same block form a set of partially 2-intersecting (ℓ, k)-partitions of the size $\binom{k\ell-2}{\ell-2} u_{\ell,k-1}$.

Corollary 5.2.3 shows that $\frac{v}{d}$ approaches a fixed constant, namely $e^{\frac{(\ell-1)^2}{2}}$, as k goes to infinity. Since the right hand side of Equation (5.21) grows linearly in k, we have that Equation (5.21) holds for k sufficiently large. This implies that if there is an eigenvalue λ of $X_{\ell,k}$ with $\lambda \leq \tau$, then the multiplicity of λ is less than or equal to $\binom{k\ell}{3} - \binom{k\ell}{2}$.

By Theorem 5.3.1, eigenspaces with dimension less than or equal to $\binom{k\ell}{3} - \binom{k\ell}{2}$ can only include the $[k\ell]$, $[k\ell - 2, 2]$ or $[k\ell - 3, 3]$-modules. The degree, d, is the eigenvalue belonging to the $[k\ell]$-module, and Lemma 5.1.1 and Lemma 5.1.2 give the eigenvalues belonging to the $[k\ell - 2, 2]$ and the $[k\ell - 3, 3]$-modules. So we conclude that $\tau = -\frac{(\ell-1)d}{\ell(k-1)}$ is the least eigenvalue of $X_{\ell,k}$ and that τ belongs only to the $[k\ell - 2, 2]$-module.

By the Delsarte-Hoffman bound, the maximum size of a coclique in $X_{\ell,k}$ is

$$\frac{|V(X_{\ell,k})|}{1 - \frac{d}{\tau}} = \frac{v}{1 - \frac{d}{-\frac{(\ell-1)d}{\ell(k-1)}}} = \frac{v}{1 + \frac{\ell(k-1)}{\ell-1}} = \frac{v(\ell - 1)}{k\ell - 1} = \binom{k\ell - 2}{\ell - 2} u_{\ell,k-1}.$$

□

The previous result verifies that the sets $S_{i,j}$ are the largest intersecting sets. We further conjecture that these sets are the only maximum intersecting sets.

Conjecture 5.3.3. *For $\ell \geq 3$ and k sufficiently large, the only sets of partially 2-intersecting (ℓ, k)-partitions with size $\binom{k\ell-2}{\ell-2} u_{\ell,k-1}$ are the sets $S_{i,j}$.*

We can make a step towards establishing this conjecture with the following weaker characterization of the maximum intersecting sets. Denote the characteristic vectors of the sets $S_{i,j}$ by $v_{i,j}$.

Corollary 5.3.4. *For a fixed integer $\ell \geq 3$ and k sufficiently large, let S be any maximum partially 2-intersecting set of (ℓ, k)-partitions. Then the characteristic vector of S is a linear combination of the vectors $v_{i,j}$.*

Proof. For $\ell \geq 3$ and k sufficiently large, $S_{i,j}$ is a maximum coclique in $X_{\ell,k}$ and equality holds in the ratio bound. Let $v_{i,j}$ be the characteristic vector of $S_{i,j}$. Since we have equality in the ratio bound, this implies that

$$v_{i,j} - \frac{\ell - 1}{k\ell - 1}\,\mathbf{1}$$

is a τ-eigenvector. Since no other modules have eigenvalue τ, these vectors are in the $[k\ell - 2, 2]$-module. Further, the set of vectors

$$\left\{ v_{i,j} - \frac{\ell - 1}{k\ell - 1}\,\mathbf{1} \mid i, j \in \{1, \ldots, k\ell\} \right\}$$

is invariant under the action of $\mathrm{Sym}(k\ell)$, so they form a module. Since the $[k\ell-2,2]$-module is irreducible, these vectors span the entire $[k\ell-2,2]$-module; this also implies that the vectors $\{v_{i,j} \mid i,j \in \{1,\dots,k\ell\}\}$ span the $[k\ell]$ and $[k\ell-2,2]$-modules.

Let S be a partially 2-intersecting set of (ℓ,k)-partitions of maximum size, and let v_S denote the characteristic vector of S. Then $v_S - \frac{\ell-1}{k\ell-1}\mathbf{1}$ is in the $[k\ell-2,2]$-module. Thus v_S is in the span of the $[k\ell]$ and $[k\ell-2,2]$-module, so v_S is a linear combination of the $v_{i,j}$. □

Note that in [24], the Meagher-Moura Conjecture has been proved for $t=2$, $\ell=3$ for all values of k.

Chapter 6

Open Problems And Further Work

In Chapter 3, we proved that the Erdős-Ko-Rado theorem holds for 2-intersecting families of perfect matchings of the complete graph K_{2k}. Our first open question is the following.

Question 6.0.1. *Can the approach we used in Chapter 3 be generalized to prove a version of the Erdős-Ko-Rado theorem for the family of t-intersecting perfect matchings of the complete graph K_{2k}, where $t > 2$? This has been done for k sufficiently large relative to t in [21].*

In this thesis, it is quite remarkable that we were able to find a weighted adjacency matrix for $M_2(2k)$ for which the ratio bound holds with equality that only uses three of the classes of the association scheme. It is an open question if a comparably simple weighted adjacency matrix would exist for larger values of t.

Question 6.0.2. *Our next question is, can we characterize the largest set of t-intersecting perfect matchings for graphs other than the complete graph? In the special case that the graph is the complete bipartite graph $K_{n,n}$, each perfect matching corresponds to a permutation. The set of intersecting permutations has been well-studied and versions of the EKR theorem hold [4, 7]. It would be interesting to consider other bipartite graphs, such as the hypercube, although just enumerating the perfect matchings may be quite difficult.*

In Chapter 4, we proved that the Erdős-Ko-Rado theorem holds for set-wise 2-intersecting perfect matchings of the complete graph K_{2k}. The very first question that arises is the following.

Question 6.0.3. *Can we extend our results to set-wise t-intersecting perfect matchings of the complete graph K_{2k} with $3 \leq t \leq \frac{k}{2}$. In Section 4.3, we defined a weighted adjacency for $t = 3$ for which the ratio bound is conjectured to hold with equality, so starting with the set-wise 3-intersecting case probably represents a good starting point for further investigation on this interesting problem.*

We also presented the following conjecture:

Conjecture 6.0.4. *The size of the largest set of set-wise t-intersecting uniform set partitions in $\mathcal{U}(\ell, k)$ with $k \geq 2t$ is $|\mathcal{S}_t(k, \ell)| = u_{\ell,t} u_{\ell,k-t}$ and a set with maximum size is an orbit of the Young subgroup $Sym(\ell t) \times Sym(\ell k - \ell t)$.*

As we mentioned in Chapter 4, the above conjecture for perfect matchings in the complete graph K_{2k} indicates that if $k \geq 2t$, the size of a largest set of set-wise t-intersecting perfect matchings is $(2t-1)!!(2k-2t-1)!!$. In Section 4.2 we proved this conjecture for $t = 2$.

In Chapter 5 we consider partially 2-intersecting partitions, but the conjecture in [23] is for partial t-intersection sets of partitions with $\ell \leq k(t-1)$. It is possible that the approach in this paper could be applied for larger values of t, but there are some steps that we predict will be complicated.

It is straight-forward to generalize the definition of $X_{\ell,k}$ to partially t-intersecting partitions by defining the graph $X_{t,\ell,k}$. This graph will also have $\mathcal{U}_{\ell,k}$ as its vertex set, and two partitions P and Q are adjacent if and only if for any pair of blocks $P_i \in P$ and $Q_j \in Q$ we have $|P_i \cap Q_j| < t$. A partially t-intersecting set of partitions is a coclique in $X_{t,\ell,k}$. Then we have the following conjecture.

Conjecture 6.0.5. *If $\ell < k(t-1)$, then the maximum cocliques in $X_{t,\ell,k}$ are exactly the canonical partially t-intersecting sets.*

Finally, while we were working on computing various entries of the character table of the perfect matching association scheme, we observed some interesting patterns for the values in the table. There are conjectures and some results about signs and values of the eigenvalues in the association scheme for the permutations, see [20]. We suspect that there are similar results for the eigenvalues in the perfect matching scheme. For example, we make the following related interesting conjectures.

Conjecture 6.0.6. *Consider the character table of the perfect matching association scheme for* $2k$*. The greatest eigenvalue in the row corresponding to the module* $[2k-2\ell, 2\ell]$ *is the one that corresponds to the same class of the scheme,* $[2k-2\ell, 2\ell]$*. In addition, in the same row all the eigenvalues corresponding to the classes which are greater than* $[2k-2\ell, 2\ell]$ *(in the dominance ordering) are negative.*

Conjecture 6.0.7. *The eigenvalues of the class* $[2k]$ *corresponding to the modules* $[2k-2i, 2i]$ *and* $[2k-2i, 2, \cdots, 2]$ *are* $-(2i-3)!!(2k-2i-2)!!$ *and* $(-1)^i(i!)(2k-2i-2)!!$*, respectively.*

Note that these two eigenvalues have high multiplicities, so we think it is worthwhile to derive their forms explicitly.

Appendix A

The Adjacency Matrices of Some Quotient Graphs in the Perfect Matching Association Scheme

In this appendix we give several quotient matrices of graphs in the perfect matching association scheme. The quotient matrix of the matrix A_λ in the association scheme with respect to the action of the Youngs subgroup $\mathrm{Sym}(\mu)$ in denoted by A_λ/μ.

A.1 The Adjacency Matrices of the Quotient Graphs Corresponding to the Group $\mathrm{Sym}(2k-2) \times \mathrm{Sym}(2)$

$\mathbf{0}$	$(2k-2)!!$
$(2k-4)!!$	$(2k-3)(2k-4)!!$

Table A.1: The adjacency matrix $A_{[2k]}/[2k-2,2]$

$(2k-4)!!$	$(k-1)(2k-4)!!$
$\frac{1}{2}(2k-4)!!$	$\frac{1}{2}(2k-1)(2k-4)!!$

Table A.2: The adjacency matrix $A_{[2k-2,2]}/[2k-2,2]$

$\mathbf{0}$	$\frac{(2k)!!}{6(2k-6)}$
$\frac{(2k)(2k-4)!!}{6(2k-6)}$	$\frac{(2k)(2k-3)(2k-4)!!}{6(2k-6)}$

Table A.3: The adjacency matrix $A_{[2k-6,6]}/[2k-2,2]$

A.2 The Adjacency Matrices of the Quotient Graphs Corresponding to the Group $\mathrm{Sym}(2k-4) \times \mathrm{Sym}(4)$

$\mathbf{0}$	$4(2k-4)!!$	$(2k-6)(2k-4)!!$
$2(2k-6)!!$	$2(5k-12)(2k-6)!!$	$(2k-6)(2k-5)(2k-6)!!$
$3(2k-6)!!$	$6(2k-5)(2k-6)!!$	$(2k-7)(2k-5)(2k-6)!!$

Table A.4: The adjacency matrix $A_{[2k]}/[2k-4,4]$

$\mathbf{0}$	$4(2k-4)!!$	$(k-4)(2k-4)!!$
$2(2k-6)!!$	$(7k-18)(2k-6)!!$	$(2k^2-11k+16)(2k-6)!!$
$3(k-4)(2k-8)!!$	$6(2k^2-11k+16)(2k-8)!!$	$(2k^2-9k+12)(2k-7)(2k-8)!!$

Table A.5: The adjacency matrix $A_{[2k-2,2]}/[2k-4,4]$

A.3 The Adjacency Matrices of the Quotient Graphs Corresponding to the Group $\mathrm{Sym}(2k-6) \times \mathrm{Sym}(6)$

0	$24(2k-6)!!$	$12(2k-8)(2k-6)!!$	$(2k-8)(2k-10)(2k-6)!!$
$8(2k-8)!!$	$8(8k-27)(2k-8)!!$	$2(13k-45)(2k-8)(2k-8)!!$	$(2k-7)(2k-8)(2k-10)(2k-8)!!$
$12(2k-8)!!$	$6(13k-45)(2k-8)!!$	$4(7k-30)(2k-7)(2k-8)!!$	$(2k-7)(2k-9)(2k-10)(2k-8)!!$
$15(2k-8)!!$	$45(2k-7)(2k-8)!!$	$15(2k-7)(2k-9)(2k-8)!!$	$(2k-7)(2k-9)(2k-11)(2k-8)!!$

Table A.6: The adjacency matrix $A_{[2k]}/[2k-6,6]$

0	$24(2k-6)!!$	$6(3k-14)(2k-6)!!$	$(k-5)(2k-12)(2k-6)!!$
$8(2k-8)!!$	$4(13k-48)(2k-8)!!$	$(34k^2-274k+564)(2k-8)!!$	$2(2k^3-27k^2+123k-190)(2k-8)!!$
$6(3k-14)(2k-10)!!$	$6(17k^2-137k+282)(2k-10)!!$	$2(32k^3-390k^2+1627k-2334)(2k-10)!!$	$(8k^4-136k^3+886k^2-2642k+3060)(2k-10)!!$
$15(k-6)(2k-10)!!$	$45(2k^2-17k+38)(2k-10)!!$	$15(4k^3-48k^2+203k-306)(2k-10)!!$	$(8^4-132k^3+838k^2-2487k+2970)(2k-10)!!$

Table A.7: The adjacency matrix $A_{[2k-2]}/[2k-6,6]$

0	$12(2k-6)!!$	$3(2k-8)(2k-6)!!$	$(k^2-7k+12)(2k-6)!!$
$4(2k-8)!!$	$10(2k-6)!!$	$(13k^2-79k+108)(2k-8)!!$	$(2k^3-21k^2+65k-52)(2k-8)!!$
$3(2k-8)!!$	$3(13k^2-79k+108)(2k-10)!!$	$(28k^3-274k^2+792k-576)(2k-10)!!$	$(4k^4-60k^3+311k^2-609k+276)(2k-10)!!$
$15(k^2-7k+12)(2k-12)!!$	$45(2k^3-21k^2+65k-52)(2k-12)!!$	$15(4k^4-60k^3+311k^2-609k+276)(2k-12)!!$	$(8k^5-164k^4+1282k^3-4591k^2+6795k-1980)(2k-12)!!$

Table A.8: The adjacency matrix $A_{[2k-4,4]}/[2k-6,6]$

$8(2k-8)!!$	$4(2k-6)!!$	$2(2k-2)(2k-6)!!$	$\frac{2}{3}(k^2-6k+2)(2k-6)!!$
$\frac{4}{3}(2k-8)!!$	$(\frac{32}{3}k-4)(2k-8)!!$	$(\frac{26}{3}k^2-\frac{116}{3}k+4)(2k-8)!!$	$(\frac{4}{3}k^3-\frac{38}{3}k^2+\frac{92}{3}k-\frac{4}{3})(2k-8)!!$
$2(2k-2)(2k-10)!!$	$(26k^2-116k+12)(2k-10)!!$	$(\frac{56}{3}k^3-164k^2+\frac{1108}{3}k-12)(2k-10)!!$	$(\frac{8}{3}k^4-\frac{112}{3}k^3+\frac{526}{3}k^2-\frac{836}{3}k+4)(2k-10)!!$
$10(k^2-6k+2)(2k-12)!!$	$60(k^3-\frac{19}{2}k^2+23k-1)(2k-12)!!$	$40(k^4-14k^3+\frac{263}{4}k^2-\frac{209}{2}k+\frac{3}{2})(2k-12)!!$	$(\frac{16}{3}k^5-104k^4+\frac{2284}{3}k^3-2486k^2+\frac{9220}{3}k-20)(2k-12)!!$

Table A.9: The adjacency matrix $A_{[2k-6,6]}/[2k-6,6]$

A.4 The Adjacency Matrices of the Quotient Graphs Corresponding to the Group $\mathrm{Sym}(2k-6)\times\mathrm{Sym}(6)$

$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$4(2k-4)!!$	$(2k-6)(2k-4)!!$
$\mathbf{0}$	$\mathbf{0}$	$(2k-4)!!$	$2(2k-6)!!$	$4(2k-5)(2k-6)!!$	$(2k-5)(2k-6)(2k-6)!!$
$\mathbf{0}$	$(2k-4)!!$	$\mathbf{0}$	$2(2k-6)!!$	$4(2k-5)(2k-6)!!$	$(2k-5)(2k-6)(2k-6)!!$
$\mathbf{0}$	$(2k-4)!!$	$(2k-4)!!$	$\mathbf{0}$	$2(2k-4)!!$	$(2k-6)(2k-4)!!$
$(2k-6)!!$	$(2k-5)(2k-6)!!$	$(2k-5)(2k-6)!!$	$(2k-6)!!$	$(6k-14)(2k-6)!!$	$(2k-5)(2k-6)(2k-6)!!$
$(2k-6)!!$	$(2k-5)(2k-6)!!$	$(2k-5)(2k-6)!!$	$2(2k-6)!!$	$4(2k-5)(2k-6)!!$	$(2k-5)(2k-7)(2k-6)!!$

Table A.10: The adjacency matrix $A_{[2k]}/[2k-4,2,2]$

$\mathbf{0}$	$(2k-4)!!$	$(2k-4)!!$	$\mathbf{0}$	$2(2k-4)!!$	$(k-4)(2k-4)!!$
$(2k-6)!!$	$(2k-5)(2k-6)!!$	$(k-3)(2k-6)!!$	$(2k-6)!!$	$2(2k-5)(2k-6)!!$	$(2k^2-11k+16)(2k-6)!!$
$(2k-6)!!$	$(k-3)(2k-6)!!$	$(2k-5)(2k-6)!!$	$(2k-6)!!$	$2(2k-5)(2k-6)!!$	$(2k^2-11k+16)(2k-6)!!$
$\mathbf{0}$	$\frac{1}{2}(2k-4)!!$	$\frac{1}{2}(2k-4)!!$	$\mathbf{0}$	$3(2k-4)!!$	$(k-4)(2k-4)!!$
$\frac{1}{2}(2k-6)!!$	$\frac{1}{2}(2k-5)(2k-6)!!$	$\frac{1}{2}(2k-5)(2k-6)!!$	$\frac{3}{2}(2k-6)!!$	$(5k-13)(2k-6)!!$	$(2k^2-11k+16)(2k-6)!!$
$(k-4)(2k-8)!!$	$(2k^2-11k+16)(2k-8)!!$	$(2k^2-11k+16)(2k-8)!!$	$(2k-8)(2k-8)!!$	$4(2k^2-11k+16)(2k-8)!!$	$(2k^2-9k+12)(2k-7)(2k-8)!!$

Table A.11: The adjacency matrix $A_{[2k-2,2]}/[2k-4,2,2]$

0	**0**	**0**	$2(2k-6)!!$	$(2k-4)!!$	$\frac{1}{4}(2k-2)!!$
0	**0**	$\frac{1}{2}k(2k-6)!!$	$\frac{1}{2}(2k-6)!!$	$(2k-1)(2k-6)!!$	$\frac{1}{2}(2k^2-7k+1)(2k-6)!!$
0	$\frac{1}{2}k(2k-6)!!$	**0**	$\frac{1}{2}(2k-6)!!$	$(2k-1)(2k-6)!!$	$\frac{1}{2}(2k^2-7k+1)(2k-6)!!$
$(2k-6)!!$	$\frac{1}{4}(2k-4)!!$	$\frac{1}{4}(2k-4)!!$	$(2k-6)!!$	$\frac{1}{2}(2k-4)!!$	$\frac{1}{4}(2k-2)!!$
$\frac{1}{4}(2k-6)!!$	$\frac{1}{4}(2k-1)(2k-6)!!$	$\frac{1}{4}(2k-1)(2k-6)!!$	$\frac{1}{4}(2k-6)!!$	$\frac{1}{2}(3k-1)(2k-6)!!$	$\frac{1}{2}(2k^2-7k+1)(2k-6)!!$
$\frac{1}{2}(k-1)(2k-8)!!$	$\frac{1}{2}(2k^2-7k+1)(2k-8)!!$	$\frac{1}{2}(2k^2-7k+1)(2k-8)!!$	$(k-1)(2k-8)!!$	$2(2k^2-7k+1)(2k-8)!!$	$(2k^3-14k^2+\frac{51}{2}k-\frac{3}{2})(2k-8)!!$

Table A.12: The adjacency matrix $A_{[2k-4,4]}/[2k-4,2,2]$

A.5 Diagonal Entries of The Adjacency Matrices of Some Quotient Graphs in the Perfect Matching Association Scheme

$(k-1)(2k-6)!!$	
	$\frac{1}{4}(2k^2+k-2)(2k-6)!!$

Table A.13: The adjacency matrix $A_{[2k-4,2,2]}/[2k-2,2]$

$(2k-6)!!$		
	$\frac{1}{4}(9k-20)(2k-6)!!$	
		$(k^3-7k^2+\frac{75}{4}k-\frac{87}{4})(2k-8)!!$

Table A.14: The adjacency matrix $A_{[2k-4,2,2]}/[2k-4,4]$

0			
	$(20k-78)(2k-8)!!$		
		$(18k^3-221k^2+953k-1455)(2k-10)!!$	
			$\frac{1}{2}(2k-11)(4k^4-60k^3+371k^2-1155kk+1530)(2k-12)!!$

Table A.15: The adjacency matrix $A_{[2k-4,2,2]}/[2k-6,6]$

Appendix B

Character Tables of the Perfect Matching Association Scheme for $2k = 8, 10$

[2,2,2,2]	[4,2,2]	[6,2]	[4,4]	[8]	
1	12	32	12	48	$\chi_{[8]}$
1	-6	8	3	-6	$\chi_{[2,2,2,2]}$
1	2	-8	7	-2	$\chi_{[4,4]}$
1	5	4	-2	-8	$\chi_{[6,2]}$
1	-1	-2	-2	4	$\chi_{[4,2,2]}$

Table B.1: Character table for $2k = 8$

[2,2,2,2,2]	[4,2,2,2]	[6,2,2]	[4,4,2]	[8,2]	[6,4]	[10]	
1	20	80	60	240	160	384	$\chi_{[10]}$
1	11	26	6	24	-20	-48	$\chi_{[8,2]}$
1	-10	20	15	-30	-20	24	$\chi_{[2,2,2,2,2]}$
1	6	-4	6	11	-26	20	$\chi_{[6,4]}$
1	3	2	-10	-8	-4	16	$\chi_{[6,2,2]}$
1	0	-10	5	10	-10	4	$\chi_{[4,4,2]}$
1	-4	2	-3	6	10	-12	$\chi_{[4,2,2,2]}$

Table B.2: Character table for $2k = 10$

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