THE INVOLUTIVE DOUBLE COSET PROPERTY FOR STRING C-GROUPS OF AFFINE TYPE

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Abstract. In this article we complete the classification of infinite affine Coxeter group types with the property that every double coset relative to the first parabolic subgroup is represented by an involution. This involutive double coset property was established earlier for the Coxeter groups of type $\tilde{C}_2$ and $\tilde{G}_2$, we complete the classification by showing it also holds for type $\tilde{F}_4$ and the types $\tilde{C}_n$ for all n. As this property is inherited by all string C-groups of these types, it follows that the corresponding abstract regular polytopes will have polyhedral realization cones.

1. Introduction

A Schurian association scheme $(G/H, G/\!\!/H)$ is the 2-orbit coherent configuration arising from the action of a finite group G on the set of left cosets G/H of a subgroup H of G. For each $g \in G$, the relation g^H is the orbital

$$g^H = G(H, gH) = \{(xH, xgH) : x \in G\}.$$

The set of orbitals is in one-to-one correspondence with the set of double cosets $G/\!\!/H = \{HgH : g \in G\}$ of H in G via

$$(xH, yH) \in g^H \iff Hx^{-1}yH = HgH.$$

This correspondence carries over to the adjacency algebra of the scheme, and indeed the elements of the standard basis of $\mathbb{C}[G/\!\!/H]$ considered as the adjacency algebra of the association scheme correspond to the normalized characteristic functions

$$\frac{1}{|H|}(HgH)^+,$$

where $(HgH)^+ = \sum_{x \in HgH} x$ for all $g \in G$. This "double coset algebra" can be identified with the Hecke algebra $e_H \mathbb{C}G e_H$, where $e_H = \frac{1}{|H|}H^+$, so the character theory of $\mathbb{C}[G/\!\!/H]$ can be understood in terms of the characters of the group G. We refer the readers to [2], [5], [8], and [15] for further information on the general structure and character theory of double coset algebras and Schurian association schemes.

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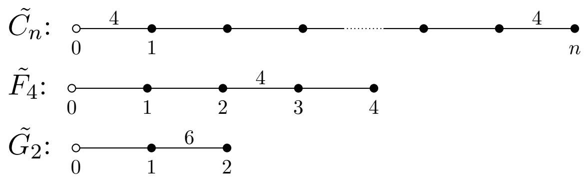
In [2], it was shown that every double coset of $G/\!\!/H$ is involutive (i.e. equal to HgH where g has order 1 or 2) in the cases where H is the vertex stabilizer subgroup (i.e. first maximal parabolic subgroup) of a string Coxeter group G that is either finite of Schläfli type not equal to $\{5,3,3\}$, or affine of types $\{4,4\}$ or $\{3,6\}$. These affine types are also identified by diagrams of type $\tilde{C}_2$ and $\tilde{G}_2$ in [10] and [12, Table 2B1]. This was then applied to show that the realization cone of any abstract regular polytopes of these types must be polyhedral.

That the involutive double coset property implies commutativity of the double coset algebra is commonly known as "Gelfand's trick", as this is one of the means of establishing the existence of a Gelfand pair. For double coset algebras of parabolic subgroups of finite Coxeter groups the equivalence of commutativity to the involutive double cosets property was noted much earlier by Iwahori; the argument appears in [4].

For infinite Coxeter groups, we will make use of the recent classification by Abramenko, Parkinson, and van Malghadem of the cases of commutativity of the double coset algebras of parabolic Hecke algebras of Coxeter groups [1]. For vertex stabilizer subgroups of infinite string Coxeter groups the results in [1] indicate commutativity for the affine Schläfli types $\tilde{C}_n$ $(n \geq 2)$, $\tilde{F}_4$ and $\tilde{G}_2$. Actually all of the effort in [1] is directed at showing other cases are not commutative - the commutativity of these cases in [1] is treated as being well-known; its proof appears to be a consequence of a theorem of [11] that makes use of special functions techniques from harmonic analysis. In Section 2 we show that the approach in [2] can be applied to show the involutive double coset property holds for the vertex stabilizer cases of type $\{4,3,\dots,3,4\}$ (type $\tilde{C}_n$) and $\{3,3,4,3\}$ (type $\tilde{F}_4$). Then we can infer from [1] that there are no other affine cases where the involutive double coset property can hold. We conclude by outlining a few consequences of our main theorem for string C-groups of these types and the corresponding regular abstract polytopes.

2. Involutive Double Cosets

Affine Coxeter groups are those Coxeter groups that have a maximal normal free abelian subgroup of finite index. The diagram for an irreducible affine Coxeter group (see [10, pg. 32] or [12, Table 3B2]) can be obtained by adding a single vertex and edge connecting it to the diagram of a Coxeter group of finite type. The *special* vertices are those that lie in the orbit of this extra vertex under diagram automorphisms of the group (i.e. automorphisms of the group that are realized by graph automorphisms of the diagram). The affine Coxeter groups whose diagrams are strings are those of type $\tilde{C}_n : \{4,3,\dots,3,4\}$ (with $n-2$ 3's), $\tilde{F}_4 : \{3,3,4,3\}$, and $\tilde{G}_2 : \{3,6\}$. Here the diagram has been oriented so that the additional special vertex is the first vertex.



We start with an immediate application of the result from [1].

Corollary 2.1. *The Schurian association schemes related to abstract regular polytopes of the following affine Schläfi types of rank ≥ 2 are always commutative: type $\tilde{C}_n : \{4, 3, \ldots, 3, 4\}$ $(n \geq 2)$, $\tilde{F}_4 : \{3, 3, 4, 3\}$, and $\tilde{G}_2 : \{3, 6\}$.*

As the only diagram automorphism of $\tilde{F}_4 = \{3, 3, 4, 3\}$ is the identity mapping, it follows from the proof of [1, Lemma 2.5] that its double coset algebra with respect to the first vertex stabilizer will be involutive. For groups of type $\tilde{C}_n$ when $n > 2$, there are non-identity diagram automorphisms. Nevertheless, for these types we have been able to adapt the direct calculation approach used in [2] for $\tilde{C}_2$ and $\tilde{G}_2$ to show the involutive double cosets property will hold.

Theorem 2.2. *Let G be a Coxeter group of type $\tilde{F}_4 = \{3, 3, 4, 3\}$. Let a, b, c, d, e be the Coxeter generators of G (in order of the diagram), and let $H_a = \langle b, c, d, e \rangle$ be the first vertex stabilizer. Then every double coset of H_a in G is involutive.*

Proof. Let $H = H_a$. According to the strategy used in [2], every double coset HwH is represented by a word of finite length that starts and ends with a, and every double coset whose representative has at least n a's is obtained by appending a reduced right H-coset repesentative beginning with and containing only one a to a double coset representative with $n-1$ a's. So the strategy is to find the left-reduced words that start with and contain only one a, using these to inductively determine the possible patterns for double coset representatives with a given number of a's, and then show each of these double cosets is equal to one represented by an involution.

For the type $\{3, 3, 4, 3\}$, the reduced H-coset representatives beginning with and containing only one a are:

$$Ha, Hab, Habc, Habcd, Habcde, Habcdc, Habcdce,$$
$$Habcdced, Habcdcedc, \text{ and } Habcdcedcb.$$

Starting with H, the double cosets that can be formed by appending these one at a time are:

$$H, HaH, H(abcdcb)aH, H(abcdcedcb)aH, \ldots, H(abcdcedcb)^{n-1}aH, \ldots$$

Since any double coset whose minimal representative contains n a's will be obtained by appending one of our left-reduced words to the representative of a double coset that contains $n-1$ a's, the above list covers all double cosets in $G /\!\!/ H$. It is easy to see that each of these double cosets is equivalent to one whose representative is an involution. ☐

It is shown in [1] that the dual Schläfli type $\{3,4,3,3\}$ has noncommutative Hecke algebra with respect to the first vertex stabilizer.

Theorem 2.3. *Let G be a Coxeter group of type $\{4,3,\ldots,3,4\}$ ($\tilde{C}_n$). Let H be a vertex stabilizer. Then $G/\!/H$ has involutive double cosets.*

Proof. The case $n=2$ is proved in [2]. For $\tilde{C}_3$, let the Coxeter generators be a,b,c,d. The reduced right H-coset representatives are:

$$Ha, Hab, Habc, Habcd, Habcdc, \text{ and } Habcdcb.$$

(The referee has pointed out it would be of interest to know the geometric significance of the set of reduced right H-coset representatives with regard to the tesselation of Euclidean space corresponding to this affine Coxeter group. We have yet to explore this direction.)

Now we repeatedly append these, starting with the trivial double coset H, and look for patterns to emerge. The first few double cosets are:

$$H, HaH, Hab\,aH, Habcdcb\,aH, Habc\,ab\,aH, \text{ and } Habcdc\,ab\,aH.$$

From this point on we only see the three infinite patterns:

$$H(abcdc)^k\,ab\,aH, (k\geq 0), \quad H(abcd)^k\,abc\,ab\,aH, (k\geq 0), \text{ and}$$
$$H(abcdcb)^k\,abcdc\,ab\,aH, (k\geq 0).$$

Again it is straightforward (though a bit tedious) to show each of these double cosets is equal to one represented by an involution. So the result holds for $\tilde{C}_3$.

For $n>3$, let the Coxeter generators be $a,b,c,\ldots,x,y,z$ (in order, so $(ab)^4=(yz)^4=1$). The reduced right H-coset representatives will follow the same pattern as in the case of $n=3$:

$$Ha, Hab, Habc, \ldots, Habc\ldots xyz,$$
$$Habc\ldots xyzy, \ldots, Habc\ldots xyzyx\ldots cb.$$

The appending pattern for the double cosets starts out in an analogous fashion to the case when $n=3$:

$$H, HaH, Hab\,aH, H(abc\ldots xyzyx\ldots cb)\,aH, H(abc)\,ab\,aH,$$
$$H(abc\ldots xyzyx\ldots dc)\,ab\,aH.$$

After this point patterns for several distinct infinite families emerge:

$$H(abc\ldots xyzyx\ldots cb)^k\,aH, (k\geq 0),$$
$$H(abc\ldots xyzyx\ldots dc)^k\,ab\,aH, (k\geq 0),$$
$$H(abc\ldots xyzyx\ldots ed)^k(abc)\,ab\,aH, (k\geq 0),$$
$$H(abc\ldots xyzyx\ldots fe)^k(abcd)\,(abc)\,ab\,aH, (k\geq 0),$$
$$\vdots$$
$$H(abc\ldots xyz)^k\,(abc\ldots xy)\,\ldots\,(abc)\,ab\,aH, (k\geq 0).$$

Again it is straightforward to verify that appending any of our left-reduced words to any of these will result in a double coset already in the list, so these are all of the double cosets. It is again straightforward to show all of these double cosets are equivalent to ones represented by an involution. The Theorem follows. □

Remark 2.4: Another consequence of [1, Theorem 2.1] is that, when G is an affine Coxeter group of type X_n other than $\tilde{C}_n$, $\tilde{F}_n$, $\tilde{G}_2$, or $\tilde{A}_1$, and H_i is a maximal parabolic subgroup of G, then $X_{n,i}$ will not be commutative. Since affine Coxeter groups contain a normal free abelian subgroup (i.e. a lattice) of finite index, G is guaranteed to have normal subgroup N of finite index (possibly a multiple of the maximal lattice) for which the double coset algebra $\mathbb{C}[(G/N)/\!\!/(NH_i/N)]$ will be noncommutative. (We are indebted to the referee here for pointing out that it is not necessary to appeal to Malcev's theorem in order to obtain a normal subgroup of finite index with this property.)

Remark 2.5: As noted in [2], a motivating consequence of our main result is that the realization cone of any finite abstract regular polytope whose automorphism group is a string C-group of type $\tilde{C}_n$, $\tilde{F}_4$, or $\tilde{G}_2$ will be polyhedral. This is because the pure realizations of the polytope correspond to irreducible characters of the double coset algebra $\bar{G}/\!\!/\bar{H}$ defined by the vertex stabilizer subgroup $\bar{H}$ of the string C-group $\bar{G}$, a finite homomorphic image of the Coxeter group of the given type. The involutive double coset property for $\bar{G}/\!\!/\bar{H}$ is inherited from $G/\!\!/H$. This implies $\bar{G}/\!\!/\bar{H}$ is a symmetric association scheme, so every irreducible character has degree 1 and is real-valued. It follows that the pure realizations of the polytope will have "essential dimension" 1 and so are the nonnegative real multiples of a single vector of length 1. The realization cone is the nonnegative real span of finitely many of these orthogonal vectors of length 1, so it will be polyhedral.

The finite regular polytopes of string affine type are the *toroidal* polytopes described in Sections 6D, 6E, 6F, and 6G of [12]. Here by the *type* of a finite regular polytope we mean that its automorphism group is a finite quotient of the Coxeter group of that type. In the *Atlas of Small Polytopes* [9], we find 18 regular polytopes of type $\tilde{G}_2$, 23 of type $\tilde{C}_2$, and 6 of type $\tilde{C}_3$. Examples of type $\tilde{F}_4$ appear in [12, Table 6E1]. For examples of type $\tilde{C}_n$, for all $n > 2$, see [12, Table 6D1].

Remark 2.6: An immediate consequence of Corollary 2.1 is that Monson's question has a positive answer for string C-groups of these Schläfli types: there can be no quaternionic pure realizations of abstract regular polytopes of these types. Cameron, Leemans, and the first author have recently found an example of a polytope of type $\{5, 5\}$ for which Monson's question, which appears as Problem 23 in [14], has a negative answer [3]; its corresponding double coset algebra has irreducible characters of quaternionic type (i.e. real Schur index 2). In light of the above results, it would be interesting to consider Monson's question for polytopes of Schläfli types $\{4,5\}$, $\{5,4\}$, $\{6,3\}$, $\{3,7\}$ or $\{3,4,3,3\}$.

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