A spectrum of possibilities: levels of improvisational behaviour in middle school mathematics

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Abstract
In this article, we consider the phenomenon of improvisation by small groups of middle years students while engaged in rich mathematical tasks in classroom settings. Working from the premise that improvisation comprises a spectrum of behaviour, we propose that there is a range of improvisational behaviours that may be observed as the students work together. We discuss four levels along the spectrum – interpretation, embellishment, variation, pure improvisation – and draw on vignettes from our research to illustrate each of their characteristics. We argue that improvisation is a valuable way to view students' mathematical performance as it highlights how students draw on their own experiences and understandings when problem solving, and how students need to be given opportunities to "stay with" mathematical tasks.

Keywords
improvisation, mathematical learning, middle years, problem-solving, small groups

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In K-12 schools, good lessons are typically described as being “planned”, good teachers as being “prepared”, and good students as being able to carefully “follow directions”. These are stereotypes, of course, but in the case of school mathematics, particularly from the upper elementary grades onwards where a designated amount of curriculum is expected to be marched through in a limited amount of time, structures like these may be very firmly in place, seemingly leaving little room for spontaneity and creativity in performing mathematics.

Awareness of the possibilities of improvisational behaviour can expand our expectations of mathematical performance. Following the work of organizational theorist Karl Weick, we argue that improvisation is not an off-on binary behaviour, where one is either following a script (off) or “winging it” (on), but rather comprises a spectrum of behaviours. We propose this spectrum as a valuable way to characterize how students perform mathematics in the classroom, one that may help educators recognize opportunities to deepen students’ understandings of mathematical concepts.

Based on the first author’s research into the problem solving behaviour of small groups working on rich tasks in middle years mathematics classrooms, this article provides vignettes to illustrate four different levels of improvisation. In doing so, we offer examples of the range of student mathematical behaviours that such tasks may permit: sometimes it is appropriate for students to “stick close to the script” in order to “get” the concepts that the task is designed to delve into; but sometimes students are able to move away from expected solution paths to make broader connections, to consider the problem situation as a whole, or to develop their own line of inquiry. If our goal as educators is to support our students in becoming more independent in their thinking about and use of mathematics, this improvisation spectrum may provide us with another way to consider our students' level of understanding of mathematical concepts.

1. Theoretical considerations

There are those who argue that our everyday lives are improvised (Bateson, 2001; Weick, 2002): our conversations and gestures (Vera & Crossan, 2004), our teamwork skills (Vera & Crossan, 2005), our ability to deal with new situations based on what we know from past situations. Writes Ryle (1979), "(t)o be thinking what he is here and now up against, [someone] must both be trying to adjust himself [sic] to just this present once-only situation *and* in doing this to be applying lessons already learned. There must be in his response a union of some Ad Hockery with some know-how" (p. 129).

Jazz music may provide a helpful model to consider how the act of improvisation works. Paul Berliner, a professor of ethnomusicology, suggests that "[i]mprovisation involves reworking precomposed material and designs in relation to unanticipated ideas conceived, shaped, and transformed under the special conditions of performance, thereby adding unique features to every creation" (1994, p. 241). The spontaneity involved in the process – the "unanticipated ideas" – is probably the characteristic for which improvisation is best known. In popular culture, to improvise something is to "just wing it." Experienced improvisers, however, know otherwise. As Berliner's definition suggests, the expertise that arises from specialised knowledge of procedures and structures associated with the craft plays an important role in improvisation: it is "the discipline and experience on which improvisers depend, and … the actual practices and processes that engage them. Improvisation depends, in fact, on thinkers having absorbed a broad base of… knowledge" (Weick, 2002, p. 51). The spontaneity experienced through improvisation is always constrained by structures that both shape and give context to free expression. Structure and spontaneity inform one another.

In jazz music, performers must have some expertise before they are able to improvise, and this comes in at least two forms: being able to physically perform the maneuvers required (i.e. the use of the instrument itself must be mastered); and having knowledge of different melodies, chord progressions, and riffs. Improvisers commonly work within forms that have certain agreed-upon fixed points, such as the reprise of a musical theme or chord structure. As jazz musician Keith Barrett writes, "Learning to play jazz is a matter of learning the theory and rules that govern musical progressions. Once integrated, these rules become tacit and amenable to complex variation and transformation, much like learning the rules of grammar and syntax as one learns to speak" (2002, p. 137).

1.1 Improvisation and school mathematics

The act of improvisation is so strongly associated with the arts, such as music, that its connections with school mathematics itself may seem at first to be tenuous. Yet there are many parallels:

> Like mathematics, music has many branches categorized in a variety of ways (classical, jazz, rock; instrumental, vocal); it has a sparse notational system for preserving information (notes, time-signatures, clefs) and theories that describe the structure of compositions (scales, patterns). However, no matter how many

> of the artifacts of music one has learned, it is not the same as doing music. It is only when one performs that one knows music. Similarly, in mathematics one can learn the concepts about numbers, how to solve equations, and so on, but that is not doing mathematics. Doing mathematics involves solving problems, abstracting, inventing, proving, and so forth. (Romberg., 1992, p. 61)

What does it mean to "do" mathematics well? According to the National Research Council (2001), there are five strands involved in mathematical proficiency – *procedural fluency, conceptual understanding, strategic competence, adaptive reasoning,* and *productive disposition* – all aspects that we believe are also required for effective improvisation.

A mathematics student needs to build experience with numerical and other patterns and relationships, and a knowledge of how (and why) to use certain algorithms, approaches, etc. in order to use these conventions effectively. This kind of automatic thinking is known as *procedural fluency,* and this is something that is strongly associated with mathematics instruction in North American society (Author). There are good reasons for this. For instance, if elementary school mathematics students are constantly struggling to work out how to multiply two numbers together, for instance, falling back on repeated addition in order to perform a calculation, it may be difficult for them to move beyond that to make connections with related concepts such as ratio and proportion.

A certain level of mastery is required to be able to improvise, and this takes practice and planning. However, while practicing leads to a freeing of the mind for jazz musicians, the danger in mathematics classes is that fluency itself may become the aim of practice. Hewitt argues that

> [p]ractice is clearly required for something new to become something which is known so well that it can be used when little or no conscious attention is given to it. However, there are many times when the carrying out of repetitive tasks through a series of questions in a traditional exercise does not succeed in helping that skill be retained beyond a relatively short period of time. (1996, p. 29)

Even when used effectively, the effect of drill exercises in school mathematics is limited, serving only to cement knowledge of procedures but not the concepts underlying them (Franke, Kazem and Battey, 2007; Hiebert, 1990; Skemp, 1976). For this reason, the next three strands of mathematical proficiency are necessary for students to make effective use of procedural knowledge. *Conceptual understanding* is needed so that the mathematics performed is actually correct and can be logically followed and proven. In both music and mathematics, then, improvisation is more about sense-making than decision-making (Weick, 2002); there is a logic inherent in the process. A student with *strategic competence* is able to identify and apply fruitful mathematical strategies, while possessing *adaptive reasoning* enables a student to switch between or combine strategies as needed, as well as to abandon ones that prove to be ineffective.

Returning to the model of jazz music, jazz musicians might be more accurately called “practicers” (Berliner, 1994, p. 494) than practitioners. The goal is “aimless aiming”:

> “[to] integrate ideas, freeing attention so that players can think strategically about their choice of notes and the overall direction of their solos. Hargreaves et al (1991, p. 53) hypothesise that when improvisers employ automatic thinking to execute patterns they are free to plan the overall strategy of the piece” (Barrett, 2002, p. 138).

To improvise effectively, then, is to be able to play with familiar structures and knowledge in order to be able to live and act in the moment, whether it be a musical one or a mathematical one.

Finally, having a *productive disposition*, in both music and mathematics, means being willing both to engage with the process and to stick with it, even when it becomes difficult, as well as to move towards something better. In his discussion of the etiquette of improvisation, not only of jazz but of actors and even of scientists solving a problem, Becker writes that people take "some elements of what they will do as given, not subject to change during the course of the improvisation, leaving [to] others as what it will be okay to vary and work with as they perform together" (2000, p. 173). In a math classroom, for instance, conventions may include explaining one's thinking if another classmate disagrees with a particular solution or helping someone who is stuck on a particular problem. These socio-mathematical norms (Cobb, Yackel & Wood, 1992) are developed over time, as group members become more familiar with one another, and these norms are always evolving, a process rather than an end product. In a more short term situation, such as solving a mathematics problem, Martin, Towers and Pirie (2006) suggest that "'better' is likely to be defined as an idea that appears to advance the group toward a solution to the problem, the drawing on a concept that seems appropriate and useful in the present situation" (p. 174). As the situation changes, this better idea evolves.

1.2 Improvisational behaviour in mathematics education

In mathematics education, several studies have focused on teachers as being improvisers and how they might be trained to take better advantage of teaching in the present moment. For example, King (2001) writes about features of musical improvisation as a metaphor for features of mathematical pedagogy, while Remillard (1999) contrasts the fixed nature of textbooks with teachers' in-action improvised decisions and responses to student needs in the classroom. Working in the realm of pre-service teacher training, Borko and Livingston (1989), Sassi and Goldsmith (1995), and Maheux and Lajoie (2010) all elaborate on ways that the concept of improvisation can serve as an analytic tool with regard to the formation of new mathematics teachers. Humphreys and Hyland (2002) see a joint emphasis on both technical skills and jazz-like improvisation as a way of countering trends towards mechanistic, positivistic teacher education that undermines teachers' professional knowledge. Ribeiro, Monteiro and Carrillo (2009) use improvisation as part of their model for teacher cognitive performance in classroom interactions, while Zazkis, Sinclair and Liljedahl (2013) argue for improvisation as the basis for pre-service teacher "role plays" that serve to demonstrate their level of understanding of how students might react to an assigned mathematical task. Towers and Martin (2009) argue for a re-visioning of the teaching role as being a part of the collective classroom unit, and suggest that preservice teachers might think about and learn about mathematics for teaching through improvisational actions such as listening to the group mind and building on the better idea.

The metaphor of improvisation has also been used to describe the possibilities of collective learning in the mathematics classroom. Neyland (2004) suggests a rethinking of mathematics instruction using jazz as a model (p. 11). He characterises a jazz combo as "a small organization that learns seamlessly as it goes along […] a thinking organization" (p. 12), and implies that students working together in small, mutually responsive learning groups with mathematical structures that are minimal enough to allow for improvisation, would afford optimal learning conditions. For instance, Francisco (2013) notes that collaborative activities give students opportunities to critically consider what makes for good arguments, and to develop more sophisticated reasoning by building on each other's ideas. Martin, Towers and Pirie (2006) and Martin and Towers (2009) take up the work of Becker (2000), Sawyer (2003) and Berliner (1994), using ideas from music and theatre improvisation to explain aspects of the growth of a fluid collective understanding in the moment that are not accounted for in longitudinal sociocultural studies of mathematics learning. Building on the work of Yackel and Cobb (1996), Martin, Towers and Pirie (2006) argue that “the lens of improvisation provides a powerful way of coordinating the analysis of the individual with the collective” (p. 154), and they characterise this collective growth as “constantly occurring, from moment to moment, and, as in an improvisational performance, is never static or constant” (p. 155).

In terms of students’ collective performance, ideas from music and theatre improvisation are also helpful. The musical idea of “striking a groove,” or interactional synchrony, is a key concept in Martin, Towers and Pirie’s work. They note that when groups of learners are involved in interactional synchrony and coaction, they appear unaware of the presence of the interviewer or the teacher, and are instead strongly focused on ideas emerging from the “collective mind” (2006, p. 173) and the pursuit of the “better idea” (2006, p. 174). This sense of the developing solution having some agency is also found in the work of Askew (2011). In analyzing the collaborative unscripted inquiry work of elementary students he worked with in London, Askew draws on the idea of “downward causation” (Campbell, 1974, cited in Askew, 2011) which in theatrical improv can be interpreted as “the scene writing itself” (Askew, 2011, p. 65). In a mathematics education context, this could be read as mathematics “playing” the mathematics learners. Askew sees this aspect of the improvisation metaphor as a conceptual opening, playing an important role in moving from either a teacher-centered or a pupil-centered lesson to a mathematics-centered one.

In this article, we seek to consider what educators might observe about students' improvisational behaviour by offering vignettes of discussion within small groups working on problem-based tasks. In the next section, we follow Weick in seeing improvisation as a spectral behaviour, and describe four different levels, suggesting how they might connect to students' mathematical behaviour while problem solving.

1.3 Full spectrum improvisation

It is not enough for a proficient improviser to deliver a jazz standard effectively, or to solve a mathematical equation, quickly and efficiently. To work entirely from a score, or a formula, striving for accuracy to the original shows only an expertise in reproduction. Improvisation requires playing with the established structures, and using some creativity, and this can occur to varying degrees, depending on how free and loose the performers are able to be. In an interview (Berliner, 1994, pp. 66-71), jazz musician Lee Konitz suggests that there exists a range of four levels of improvisational behaviors, depending on the proportion of structure to spontaneity: Interpretation, Embellishment, Variation and Improvisation. (In this article, we will refer to this last one as "Pure Improvisation" to distinguish it from improvisational behavior in general). The existence of a range implies that improvisation is not a binary phenomenon, where there is a state of being either "off" or "on" in terms of whether one is improvising or not. Weick builds on Konitz's proposal by arguing for a "full spectrum" of improvisation. This "has different properties than stand alone [improvisation] and makes fuller use of memory and past experience, can build on the competencies of a more diverse population, is more focused by a melody, and may be more coherent" (2002, p. 53), qualities that are also very suitable for describing the situation in a classroom situation (with a mathematical model supplying the melody). In this section, we will explore these properties further by considering each of Konitz's levels in terms of what an observer might notice both in a jazz performance, and what an observer might notice in the performance of a classroom mathematics task.

Full spectrum improvisation runs from one end (Interpretation) where there is a high dependence on established models, be they scripts, scores, riffs or mathematical algorithms, to the other end (Pure Improvisation) where there is a high degree of independence from these models. As one moves along the spectrum from Interpretation towards Pure Improvisation, there the distance between the structure of the problem and the students' performance increases and thus there becomes a greater amount of room in which to "play." There are many ways to define the word "play" but what we have in mind is the image of a person who is fishing allowing "play" in her line rather than keeping it taut. A line that has play in it enjoys "freedom of movement" or the "space or scope for this [movement]" (Barber, 1998, p. 1112). The less students rely on the structure of the mathematical task, more freedom they have to play, and the more room they have to include their own experiences and ideas in their consideration of the task. Following Konitz, we will now describe four levels of improvisational behaviours.

Interpretation

In the full spectrum model, Interpretation is the level where the performers are most dependent on pre-existing models. In jazz, for instance, this would occur when a song is performed, with the musicians playing all the notes, in the original order, but varying musical features like attack, stress, tempo and dynamics in order to create their own interpretation. The original song is intact, and very recognizable, but the approach of the musicians gives the performance of the song a unique style that would distinguish it from the performance of the same song by a different group of artists. In the mathematics classroom, Interpretation would occur when students are addressing a task by following the prescribed steps but performing them in a way that shows the nature of their understanding. For instance, students may be working on a structured task such as finding the perimeter of a rectangle, where there is a single end point (or correct answer), and a limited number of mathematically correct ways to approach the answer, such as adding the lengths of the sides together, or perhaps multiplying each of the measurements for width and length by two and then adding these products together. Although there is not much room for mathematical variety in developing such a solution, it is our experience that teachers will still notices some differences in how their students report their solutions. Some might use drawings to show their ideas, others may write down calculations, still others may create a story narrative, and so on.

Embellishment

With Embellishment, there is further movement by the performers away from pre-existing models. For instance, when jazz artists play a certain song it is recognizable as being that song but they are not playing all of the original notes in their original sequence. Instead, the musicians rephrase the melody by moving parts around to anticipate or delay them, or they may add ornamentation, or unexpected chord sequences. The original song structure is still present, but the musicians have embellished it with their changes. In a mathematics classroom, Embellishment may occur when students bring in experiences that are closely related to the mathematics task, such as concepts or strategies that they have explored in previous mathematics tasks that they believe might be applicable. Although the students may ultimately end up following a typical path towards a solution, the stops they make along the way as they connect the task with related experiences out provide embellishments that enrich their solution. We consider this level of the spectrum to involve an act of productive inquiry, “where we are deliberately (though not always consciously) seeking what is we need, in order to do what we want to do” (Cook & Brown 1999, p. 388).

Variation

In Variation, there is further distance between the pre-existing model and the act of performing it. In jazz, this occurs when the song is played, but the musicians alter the song by inserting groups of new notes into it. Although clearly these notes are related to the song, they also offer a side-exploration of musical themes. In a mathematics classroom, episodes of Variation may happen when the students find one way to model a task but then decide to explore other models further in order to find an additional, and potentially more interesting or appropriate, solution pathway. Here an educator would observe students considering the task situation as a whole and sharing and trying possible strategies that are not obviously related to the task at hand but might still be effective.

Pure Improvisation

At the Pure Improvisation end of the spectrum lies the potential for transformation, creation and discovery (Weick, 2002). In jazz, the song structure provides a starting point but parts of the melody are changed, or completely replaced, to such an extent that the resulting song does not resemble the original. In mathematics class, students might start with the original problem task, but pose a series of new problems until the ideas they are exploring and inventing are far from their starting point. It is only when these mathematical performers look back that they can discern a pattern to their actions and ideas. Or students may begin with a problem that is entirely their own and explore other mathematical ideas from there. In either case, the students are demonstrating mathematical independence.

2. Methodology

Each of the research excerpts that we will discuss took place in grade 8 mathematics classrooms in a middle school in large suburban school district in Western Canada. This age group is known for its high energy and for its enthusiasm for socializing, making its members well-suited for working in groups while tackling mathematics tasks. The vignettes that we use to illustrate the Interpretation, Variation and Pure Improvisation levels are drawn from research about Mrs. Shug's mathematics classes, and the vignette used for the Embellishment level is from one of the researcher's own mathematics classes (first author). The vignettes all involve students working in designated small groups on rich mathematics tasks in their mathematics classrooms while being recorded by an unattended stationary video camera. All students were considered by their teachers to be at grade level in terms of their mathematical ability, and all were confident enough in their own mathematical performance to agree to participate in the studies. To preserve their anonymity, pseudonyms have been used for the teacher and students referred to in this article.

In jazz, "melodies vary in the ease with which they evoke prior experience and possibilities than do other melodies" (Weick, 2002, p. 54). This idea is echoed in the mathematics education literature in the idea that some problems are more "rich" or conducive to triggering student thinking. According to one popular mathematics education textbook for Canadian pre-service teachers:

> Rich tasks offer boundaries or constraints within which students have the freedom to explore. Rich tasks offer the opportunity to transform student thinking, make new connections in their conceptual network, expand their ability to act in a mathematical space, and to notice what they had not previously noticed. Tasks should employ mathematical processes (e.g. reasoning, problem solving, and communication) and help cultivate dispositions of inquiry" (Van de Walle, Karp, Bay-Williams, McGarvey & Folk, 2015, p. 30).

According to Small (2009), rich tasks can take a variety of forms – open ended prompts, problem solving, problem posing, words, symbols, a few minutes or days/weeks to investigate, model or abstract, and so on – and may have the following features:

- variable entry and exits (Simmt, 2001);
- opportunities for mathematical thinking (Mark, Cuoco, Goldenberg & Sword, 2010);
- high levels of cognitive demand (Smith & Stein, 1998);
- relevant contexts.

To consider the nature of the mathematical tasks given to the students in our research, we use a framework proposed by Smith and Stein (1998). The two tasks ("Bill Nye Fan Club" and "The Locker Problem") used in the vignettes described in this article each have a correct answer but no prescribed pathway for a solution. As such, these tasks offer what Smith and Stein describe as higher-level demands ("doing mathematics"):

- require complex and non-algorithmic thinking – a predictable, well-rehearsed approach or pathway is not explicitly suggested by the task, task instructions, or a worked-out example;
- require students to explore and understand the nature of mathematical concepts, processes, or relationships;
- demand self-monitoring or self-regulation of one's own cognitive processes;
- require students to access relevant knowledge and experiences and make appropriate use of them in working through the task;
- require students to analyse the task and actively examine task constraints that may limit possible solution strategies and solutions;
- require considerable cognitive effort and may involve some level of anxiety for the student because of the un- predictable nature of the solution process required (1998, p. 348).

In his discussion of improvisation, Neyland states that jazz players seek an "optimally minimal structure that allows for the best creative improvisation. Too little or too much structure stifles creative improvisation" (p. 12). The balance between structure and improvisation, or *disciplined improvisation* (Sawyer, 2004, 2011), has been explored by studies that have considered the implications of this in the classroom (King, 2001; Sawyer, 2011). We believe that the rich tasks used in these vignettes offered an optimum structure that allowed for a range of improvisational behaviours.

3. Findings

In the following section, we present a vignette for each of the levels of the improvisation spectrum. In the transcript excerpts, [ ] indicates a description of what was happening. /   / indicates overlapping speech, and "…" indicates a pause of three or more seconds.

3.1 Interpretation

This vignette focusses on a group of four students working on the following task:

> **The Bill Nye Fan Club is having a year-end party, which features wearing lab coats and safety glasses, watching videos and singing loudly, and making things explode. As well, members of the club bring presents to give to the other members of the club. Every club member brings the same number of gifts to the party.**
> **If the presents are opened in 5 minute intervals, starting at 1:00 pm, the last gift will be opened starting at 5:35 pm. How many club members are there?**

In our experience using this task with students, students first need to figure out the number of time intervals (56). Then students need to determine how many club members bring how many gifts (the factor pair 8 and 7). The wording of this task is somewhat ambiguous, so in her introduction to the task Mrs. Shug let the class know that club members do not bring presents for themselves. This limits the answer to the pair of factors where (n)(n-1) = 56. Middle year level students might determine the number of club members (8) through guessing and checking, or systematically going through the factor pairs of 56.

After much discussion to determine the correct factor pair, three members of a small group, Derek, Amaya and Timothy, began to talk in more detail about ways in which one could determine the number of intervals.

Derek: Why did we get 56 in the first place?

Amaya: Because here *[Amaya refers to the calculations she has written down in front of her that show she was using the idea of a 24 hour clock]* everything started at thirteen. One turned to twenty-four hour clock is thirteen. Exactly.

Derek: Ah.

Amaya: And then everything ends at 17:35 because five into twenty-four hours equals seventeen and then minus them is how many hours it took to do the entire thing

[Derek begins counting while she speaks – he appears to be using the clock on the classroom wall near the group for his counting just as Timothy had been doing earlier in the session in order to determine the amount of time]

Derek: Wow, that's a waste. You didn't have to do all that. You could have just looked at the clock.

Timothy: I know. That's what I did.

Amaya: (*inaudible*).

Timothy: It goes like this. You don't have to do all that.

Amaya: How do you do that?

Derek: As you can see, you start at the – where is it, where is it – the one. One. *[All three students are now looking at the wall clock]*

Amaya: Yeah.

Derek: What time is it?

Amaya: 5:35.

Derek: One. Four hours and four minutes. That's how long it took because the last present starts [being opened] at 5:35.

Amaya: Holy crap.

Derek: What?

Amaya: How do you do that? Like the clock how many hours?

Derek: Well anyways. Going back.

Amaya: *(inaudible)* four 40 *(inaudible).*

Derek: That's how many. Yeah, yeah, yeah.

Amaya: Four forty. Oh yeah. Four times, yeah, the hours thingie. Four times 15 equal to

Derek: Man, your work is *(inaudible).*

Amaya: *[laughs]*

Timothy: Yeah, it's 56. Just count around the clock.

There were four groups in this particular mathematics classroom who were recorded working together on the Bill Nye task and all followed the same basic solution path of finding the number of time intervals and then determining the appropriate factor pair for the task situation. However, this was the only group to discuss how to use the 24 hour clock and how to physically count around on an analog clock to determine the number of intervals. This use of alternate methods for a small portion of the task is what makes this an example of Interpretation. Although the group members have taken a detour to explore ways to determine the number of intervals, they are still following a common solution pathway for the task. This particular detour was likely prompted by what was available to students, an analog clock that was physically near them on the classroom wall, and experience gained during a previous task from the same research study that used a 24 hour clock system.

3.2 Embellishment

The Locker Problem is a task that has been adapted for use with students ranging from middle school age to university level and is one that can be found in many math resources. The problem sheet that the group was given read as follows:

> **A newly built school has 1000 lockers, numbered 1 to 1000, located down one side of a really really long hall. The lockers are unlocked and the doors are shut. A student walks down the hall and opens the doors to all 1000 lockers. A second student follows and closes the doors of the even numbered lockers. A third student changes the position of the doors numbered 3, 6, 9, 12,…. (that is, opens closed doors and closes open doors). A fourth student changes the position of doors 4, 8, 12, 16,…. This process continues until the thousandth student changes the position of the door of the locker numbered 1000. Which doors are open?**

In our experience, middle school students typically tackle this problem by monitoring what happens to each locker door as each "student" in the problem walks down the hall and looking for a pattern. They might do this through a drawing, a chart, or by physically acting it out with lockers in their hallway. Students may try to figure out what is happening to all 1000 lockers or choose a more manageable set of the lockers with which to work. The pattern that eventually emerges that reveals that all the open lockers have numbers – 1, 4, 9, 16, 25, and so on – that are square. As the lockers doors begin by all being closed before the first "student" goes by, only lockers whose doors are handled an odd number of times will end up being open, namely those lockers whose numbers have an odd number of factors (square numbers).

The group of four girls in this vignette (Anik, Sandi, Khona and Crystal) start off by working with the information provided by their problem sheet. First they focus on which students changed the door positions of which particular lockers.

Anik: First student opens <u>all</u> of them.

Sandi: Yeah, so he opened all of them.

Unknown: Second student.

Unknown: I think I got it.

Unknown: Closed all of the uneven ones.

Sandi: All of the even ones.

Khona: <u>Un</u>even ones.

Unknown: No he closed the even ones.

Sandi: Then the third one.

Khona: Oh.

Sandi: Um, numbers 3, 6, 9, 12, you know up like that, if they were open he'd close them, and if it were closed he'd open it, whatever those numbers are

Anik: Like the 3?

Sandi: Yeah… multiples of 3.

Sandi: Yeah. And then fourth student multiples of 4 change the position to either…

Anik: And then there'd be the multiples of 5.

Sandi: Yeah.

Anik: 6.

Sandi: And the multiples of every other number, up to 1000.

In their subsequent discussion, the group begins searching for a pattern in the data in order to solve the problem, keeping track of which doors were opened by which student, although this was not explicitly mentioned until later in the discussion. At first the girls focus on the larger idea of "multiples" (a term which they were using to mean "factors") until a more specific idea emerges:

Anik: Wait wait I know what we can do (*inaudible*) we know (*inaudible*) don't have any (*inaudible*) Like 17, you can't times any number to get 17 (*inaudible*) is that right?

Sandi: Yeah.

Anik: So 17 would always be open. So we can use that to figure out which ones will always be open.

Sandi: So where do we start?

Anik: I'm not quite sure.

[*pause*]

Sandi: So those numbers will stay closed.

Anik: No it stays open.

Sandi: So the first one opens all of them, and when it gets to their number they open that one so it'd be closed.

Khona: Prime numbers, are they called prime numbers?

Unknown: Yeah the ones that can't.

Anik: Oh, I can't remember.

At first the group's discussion focuses on the name "prime numbers" and the properties of prime numbers, in particular how these numbers each only have two factors. Then they turn to other resources.

Khona: How many prime numbers are there?

Crystal: I know, we have the sheet.

The sheet Crystal refers to was from a class activity four months earlier which modeled the Sieve of Eratosthenes as a way of determining all the prime numbers between 1 and 100. In this activity, each of the numbers in this range was systematically checked to see if other numbers could divide evenly into them. Those numbers which could only be divided by 1 and themselves – for example, 17 can only be divided by 1 and 17 – were identified as prime numbers. The girls attempt to use this sheet as a basis for finding a pattern for determining how many prime numbers there were between 1 and 1000. When there does not seem to be a visible pattern between 1 and 100, they decide to break the range of numbers up into groups of 10 (i.e., 1 – 10, 11 – 20, etc.) to see if they might be able to locate any patterns that way. Then Anik offers another suggestion.

Anik: Wait, I just figured something out, look.

Sandi: What?

Anik: If you figure out how many times any number… 1000 goes into each number, like, Say you pick the number 20. You know 2 can go in it, into it, 4 can go into it, 5 can go into it, 10 can go into it, and if I'm right that's it. So you know four numbers can go into it.

Sandi: Right.

Anik: It starts open, or it starts closed, so it will end closed. Or, I mean it will end open.

Sandi: Oh my god, you're so smart Anik.

Anik: Closed, open, closed, open.

Sandi: Yeah.

Anik: It will end open. And if it's odd then it will end the same as it started. So it will be even if the number of numbers that goes into it is, I don't know if I'm making any sense.

Sandi: Okay, you know what? You're smart.

After more discussion, the entire group accepts this new idea, and for the few minutes remaining in the session the girls used a multiplication chart to help them identify how many factors certain numbers had. Although the girls ran out of time before they have a chance to identify that what they are looking for is square numbers (i.e. numbers which have an odd amount of factors), they appear to be moving in that direction – in their written assignments that are later individually submitted for homework both Anik and Sandi show that they have each eventually figured it out independently.

In solving this problem, this group of girls took a mathematically rich journey, drawing on their previous mathematical experiences in their discussions of how to approach this task and exploring related mathematical concepts in the process. They carefully considered the nature of fractional numbers (an idea proposed early in the session), multiples, factors, and prime numbers, all topics they had studied in their mathematics class earlier in the school year. In the course of their discussion, the girls refer to resource sheets in their binders that were from previous class activities, in particular, a multiplication chart that had been used for a multiples/factors task, and the Sieve of Eratosthenes sheet that had been used for a prime numbers task. The girls also used problem solving strategies such as seeking important information in the given task, using diagrams, finding a pattern, and breaking up sets of data into more manageable chunks. These were all strategies that had been discussed on "problem solving days" in their previous mathematics classes, and the girls often explicitly mentioned what they are about to do (for example, Anik calling for a felt pen so she could "highlight all the useful information"; Sandi stating that she would normally write down all the [locker] numbers but she was only going to do a set of 10 to begin with; frequent references to looking for the pattern) and how their teacher would not be expecting them to write down all 1000 numbers. In drawing on all of this, the girls embellish and enrich their developing solution path with discussions about the mathematical concepts and strategies. It is discussions like these that mark this group as performing at the Embellishment level of the improvisation spectrum.

3.3 Variation

When introducing the Bill Nye task (introduced earlier in this article in the Interpretation vignette) to her class, Mrs. Shug appears to anticipate that her students might have difficulty with the term "interval" in the task and asks the class what it means. Ian, from one group, offers the following suggestion: "So one minute, so you'd open the present, then five minutes later you open another one," a definition that Mrs. Shug accepts and expands to "Okay? So it's a time that's a base time. Okay?"

As the members of the group for this vignette (Rebekkah, Eliana, Geri and Lucy) start to work together, it quickly becomes apparent that despite this class discussion they are still not sure what an interval is.

Eliana:	I don't get the five minutes, the five minute intervals. How does that work?
Rebekkah:	Like to break for five minutes
Geri:	What's a tournament?
Eliana:	Tournament?
Geri:	Is it two and one goes out?
Eliana:	Yeah.

In asking "What's a tournament?" Geri seems to be suggesting the model that two teams, or players, compete and whoever is defeated leaves the competition. Apparently, the group does not find this helpful for considering the logistics of the gift-giving, and the model of a tournament is dropped. After further discussion within the group about the task in general, the question of how to define an interval reemerges.

Rebekkah: So I guess we have to do, say each gift takes a minute to open, we have to do like a minute then five minutes then a minute, five minutes, until it gets all the way 'til five thirty-five.

Eliana: Wasn't five, wasn't five minutes the time to open the present though?

Rebekkah: That's the interval. So that's the break after each present's opened. It's like, okay, so I open up a gift and then we go have some food for five minutes and then go back and open up another one.

Rebekkah's model of the interval being a break for food between opening gifts does not appear to be helpful either, and the group continues to be stalled.

Eliana: Intervals, I need to make it the intervals clearer I think.

Rebekkah: 'Cause like, it's just a break like.

Eliana: 'Cause it's like [Ian] was like the time to open the presents I think. I thought that's what [Ian] said.

Rebekkah: When I think of intervals I think of like plays and there's all these like intervals and people stop for twenty minutes to have a snack or something

Lucy: I thought it was intermission.

Rebekkah: Oh right. Well they both sound the same, they are the same – intervals, intermission. Whatever.

There appears to be a difference of opinion as to what Ian meant during the earlier general class discussion. While Eliana suggests that Ian defined "interval" as the time taken to open each present, Rebekkah seems to be following his words more literally – one minute to open the gift and then five minutes to wait before opening the next one. Given that, realistically, it does not take five minutes to physically unwrap a gift, this second interpretation would be the most likely of the two. Rebekkah repeats her earlier suggestion of "a break" and then developed this model further with the description of it being like when people attend a play and "stop for twenty minutes to have a snack." Lucy suggests that what Rebekkah actually was describing was called an "intermission." At this point, they call Mrs. Shug over and she clarifies that it takes five minutes to open one gift.

Once they have a better idea of what an interval is, the group members are able to determine quite quickly that they have 56 intervals of 5 minutes to work with, and that solving the task will involve finding an appropriate factor pair of 56. Yet they seem unable to home in on which factor pair it is, reviewing the pairs of 1 and 56, 2 and 28, 4 and 14, and finding that none of them work for the task situation: there is something about these pairs that, as Rebekkah points out, "doesn't make sense." Two people giving each other 28 gifts ws just silly, and while 28 people giving out two gifts each may not be as extreme, it does not fit the task situation either because it would mean that club members are not giving gifts to every other club member.

Earlier in the session, Geri had been trying to remember if she had ever worked on a task like this before. Now she offers a suggestion:

Geri: [still writing] Yeah it's 55 right? [stops writing] And then see if there's ten people? So it's different than these people then this person so they give out one, two, three, four, five, six, seven, eight, nine, ten presents. Ten. And the second person gives out one, two, three, four, five, six, seven, eight, nine because you're already given.

The diagram that Geri offers the group is one that might be used to illustrate the classic "handshake problem" where everyone in a group shakes hands with each other exactly once, with lines drawn from each "person" (symbolised in Geri's drawing as a little circle) to all the other "people" with whom they shake hands.

Eliana: But everyone has to give. But everyone has to give the same amount of presents.

Geri: Yeah.

Rebekkah: Yeah.

Geri: But like isn't it like [an] exchange?

Rebekkah: No it's like giving out.

Eliana: Because, okay, every club member brings the same number of gifts to the party.

Rebekkah: So it's always.

Geri: But it doesn't make sense.

Unknown: It doesn't make sense.

Rebekkah: I'm sure there's more to it.

Eliana: Yeah I that was a good theory. Omigod I wish we could have done something cool like that so we could show it.

Rebekkah: Yeah I know, we could do lines everywhere and what.

Geri's group members appreciate her work and diagram, particularly considering Mrs. Shug's earlier recommendations to her students to "think outside the box" and to communicate their thinking clearly, but Geri's model of an exchange does not fit the task situation either. Again, it does not make sense. However, Eliana and Rebekkah's explanation as to why it does not make sense (every club member having to bring the same number of gifts) appears to prompt a discussion of who could give to whom and how many presents they could give. This seems to help focus the group's understanding of what is going on in the task situation as, shortly after this, they finally hit on the one pair of factors of 56 that had somehow been eluding them, 7 and 8.

Eliana: Hey yeah eight eight eight times seven right? Cause eight people and eight per eight people [excited]

Unknown: /And seven/

Unknown: /Seven. So they have/

Rebekkah: You don't give to yourself right?

Eliana: Yeah exactly. Omigosh. That's good

Rebekkah: 'Kay

Eliana: I think. Does it work?

Rebekkah: Yeah it works. So there's 8 people.

Eliana. Eight.

Rebekkah: Omigosh. It makes total sense now.

In this session, the girls are using different models to get an overall sense of the logistics of the gift exchange in the task, in particular what is happening during the intervals. They consider the models of a tournament, a break or intermission, and an exchange until finally they hit upon a model (which remains unnamed) that works for them. It's their exploration of different ways of considering the task situation as a whole that marks these episodes of their work as being in the Variation stage.

3.4 Pure improvisation

The following short vignette took place with another of the four groups in Mrs. Shug's Grade 8 class that had been working on the Bill Nye task. This group (Nitara, Ian, Jacob, Michael) has finished working on the task and reached the correct answer. One member of the group, Ian, is double-checking the group's calculations while the rest of the group await his results. Ian has just suggested that the group needs to follow Mrs. Shug's suggestion to everyone at the beginning of the class to work "outside the box" in order to determine whether or not their proposed solution is correct, joking that one cannot get more outside the box than group member Michael is. In response, Michael initiates the following discussion:

Michael: I wonder what a triangle would look like in real life…. So, you see a triangle on paper, right?

Nitara: Yeah.

Michael: I wonder what it would look like if you could pick it up.

Nitara: It would look like one of those really cool Toberone bars. When you break them in half there's a triangle.

Michael: Doesn't it start out as a triangle?

Nitara: No, it starts out as a like long triangle.

Michael: Prism.

Nitara: No, a prism is a… yeah.

Jacob: Yeah, it is a prism.

Michael: I think it is a prism.

Nitara: A long prism.

Michael: Well, that's what a prism is.

Jacob: Why are we talking about triangles?

Michael: It's an extended triangle I think. See, I can be smart.

Jacob: *Could* be.

This problem of "what would a triangle be like in real life" was student-generated, a response to the challenge of mathematically thinking outside the box, and the discussion that followed was also emergent – there was no predicting where the students might have gone with this. As well, there was no designated end point – the conversation was an end in itself. The discussion was brief, whimsical, but most importantly, it was mathematical, and there seems to be a distinct pleasure taken by Michael in working well beyond the boundaries of the prescribed task yet still doing something that his group recognised as being mathematics. This marks this vignette as being an example of Pure Improvisation.

4. Discussion

The conception of mathematical improvisation as a spectral behaviour offers educators an opportunity to re-envision how students' engage with mathematical problem tasks. In the following section we will discuss the implications of past experiences and "staying with" a task on improvisational behaviour.

4.1 Past experiences

As students move further towards the Pure Improvisation end of the spectrum, the levels of the spectrum become "more heavily influenced by past experience, dispositions, and local conditions" (Weick, 2002, p. 52). Two of the groups draw on their recent experiences in their mathematics classes. In the vignette for the Interpretation level, although the group sticks quite closely to the structures of the task, towards the end of their work together, three of the students discuss current (the analog clock that is physically near them in their classroom) and very recent experiences (a mathematics task from a previous week that involved the use of the 24 hour clock) that have influenced their interpretation of how calculate the number of intervals. The girls working on the Locker task (Embellishment level) similarly draw on their experiences with, and resource materials from, previous activities in the school year in their rich, and extended discussion about the nature of fractions, factors, multiples and prime numbers. In doing so, they develop a solution path for their task that features frequent stops along the way which serve to add to, or embellish, their performance of the task, an illustration of how a small group may pursue the better idea and how their idea of "better" evolves as the group's thinking develops (Martin, Towers & Pirie, 2006; Becker, 2000).

The other two groups, who perform further along the improvisational spectrum, draw on experiences outside of the mathematics classroom. Eliana, Rebekkah, Grace and Louise (Variation) suggest different real-life situations as ways of modelling what is happening with the gift-giving and intervals in the Bill Nye club task. During their discussion they consider, and then reject, the ideas of a tournament, intermissions during a play, and an "exchange," before working an unnamed model that helps them to grasp what is going on and finally reach a viable solution. Finally, although it may seem unlikely that an episode of Pure Improvisation could occur in a classroom situation where students are directed to solve a specific task, this excerpt shows that it is possible. This group's discussion of the triangle is a good example of Neyland's idea of "playing outside" (2004). In pondering what a triangle might look like in "real life," the group considers the relationship between a triangle, a two-dimensional shape, and a prism, a three-dimensional solid. First, a real-life example is proposed, explaining how, if a Toberone chocolate bar broken in half, the end's surface face will be revealed to be a triangle shape. The students have a short debate over which comes first, the triangle or the prism. Then they characterize the prism as a "long triangle" or an "extended triangle," a nice way of highlighting the height dimension found in a prism that a drawing of a two-dimensional triangle would lack. It is a brief discussion, but a mathematically rich one.

It is important to note that we are not suggesting that one level of improvisational behaviour is better or more desirable than another. Although it is beyond the scope of this article, in the case of the group who reached the Pure Improvisation level, just prior to their discussion about what a "real" triangle would be like they had been working at the Variation level, using another model to double check their solution. As groups work on a mathematics task, it is likely that they move around along the improvisation spectrum in terms of how they interact with the task's structures and the type of experiences they bring to their consideration of the task, and thus different levels of improvisation may serve different needs. This is a behavior that Weick has noted in the performances of jazz groups (2002), and one that would be worth further research in mathematics classrooms.

4.2 Staying with the situation

As Van de Walle et al (2015) point out, it is important that students spend time in considering the mathematical tasks on which they are working: "The more students stay with and continue to explore a mathematical situation, the more opportunities they have to learn" (p. 26). In all four of the vignettes, the students "stayed with" the task situation to consider it further. The group in the Interpretation vignette and the group in the Pure Improvisation vignette both stayed with their tasks after they had found answers so that they could review their work. In the Variation vignette, Amaya, Derek and Timothy revisit how to determine the number of time intervals and compare two possible methods, doing calculations using 24 hour time and physically counting around the classroom clock. In the Pure Improvisation vignette, it is while Ian is checking their solution for the Bill Nye that the rest of the group has an "outside the box" discussion of how a triangle would appear in real life. The other two groups, at the Embellishment and Variation levels, realise that they are stuck and both spend time discussing various mathematical ideas to try to get themselves back on track. All the vignettes illustrate how students need time to check and reflect on their solutions, to think critically about their mathematical understanding, to consolidate what they have learned, and to extend their thinking. This is an important reminder that there needs to be both time available for students to engage in this type of thinking, and a safe classroom culture in place that encourages students to do so.

5. Conclusion

We hope that the concept of a spectrum of improvisation may offer an alternate way to view mathematical behavior in the classroom, pointing to the possibilities for students to develop deeper levels of understanding when they are able to draw on and work with their own experiences and understanding, and to stay with mathematical tasks for a meaningful length of time. In education systems where efficiency is prized as a way to get the mathematical curriculum “covered,” this is something that is often sacrificed. Perhaps an improvisational lens may help us to reimagine a whole spectrum of other possibilities for our mathematics students.

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